Actuality, Tableaux, and Two-Dimensional Modal Logics

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Abstract

In this paper we present tableau methods for two-dimensional modal logics. Although models for such logics are well known, proof systems remain rather unexplored as most of their developments have been purely axiomatic. The logics herein considered contain first-order quantifiers with identity, and all the formulas in the language are doubly-indexed in the proof systems, with the upper indices intuitively representing the actual or reference worlds, and the lower indices representing worlds of evaluation — first and second dimensions, respectively. The tableaux modulate over different notions of validity such as local, general, and diagonal, besides being general enough for several two-dimensional logics proposed in the literature. We also motivate the introduction of a new operator into two-dimensional languages and explore some of the philosophical questions raised by it concerning the relations there are between actuality, necessity, and the a priori, that seem to undermine traditional intuitive interpretations of two-dimensional operators.

1 Introduction

In this paper we present semantic tableaux for two-dimensional modal logics. First we devise tableaux for a system we call $\mathbf{S5}_{2D}$. The logic is somewhat based on the systems presented in both Crossley and Humberstone (1977) and Davies and Humberstone (1980), but it extends these significantly. The language contains the usual Boolean connectives, modal operators, first-order quantifiers, identity, an actuality operator, a *Ref* operator, a fixedly operator, and an extra actuality operator that we call "distinguishedly". All the formulas in the language are doubly-indexed, with the upper indices intuitively representing the actual or reference worlds, and the lower indices representing worlds of evaluation — first and second dimensions, respectively. Additionally, we show the tableaux to be modular in the sense that they can be adjusted for different notions of validity: local, general, or diagonal. We show the tableau methods for $\mathbf{S5}_{2D}$ to be general enough for different logics such as the $\mathbf{S5}$ system with an actuality operator presented in both Crossley and Humberstone (1977) and Hazen (1978), the logic for the fixedly operator developed in Davies and Humberstone (1980), as well as variations of a different system containing the so-called diagonal or apriority operators as primitive, which is presented in both Restall (2012) and Fritz (2013, 2014).

Models for such logics are well known, as there are several axiomatic systems for different two-dimensional logics.[1] By contrast, different proof systems have been rather unexplored. Besides the axiomatic developments we note only the hypersequent system introduced by Restall (2012) containing apriority operators.[2] Furthermore, the great majority of axiomatic systems — and Restall's hypersequent system — are presented exclusively for the propositional case.[3] This is because the main innovations are engendered by the two-dimensional operators themselves, whence not much appears to be gained by adding first-order quantifiers, especially given the notorious complications they bring into modal contexts. However, the motivation for the introduction of two-dimensional operators is usually drawn from examples in a first-order language, as illustrated by (1) (and (3)) below, and, moreover, there are several interesting questions concerning names, identity, and rigidity, to name a few, once we take quantification into account in logics containing both modal and epistemic operators.[4] We shall explore some of these questions in due course.

Moreover, no tableau system seems to have been developed thus far for two-dimensional logics.[5] One advantage of this approach — and, as argued in Ray and Lampert (manuscript), also natural deduction systems — for modal logics is that tableaux containing numeric indices can be construed in a quite natural and elegant manner, where the indices refer intuitively to possible worlds. This provides a certain semantic transparency to the proof system that becomes manifest in the rules, for they basically display the semantics for each operator.[6] Another advantage is, or course, its practicality. It is very simple to prove theorems in it, whereas the same is not true for axiomatic systems. And, in effect, in order to develop two-dimensional tableaux we can assume much of what is already familiar regarding tableau techniques for modal logic, which is more of a virtue than anything else; as two-dimensional modal logic tends to be at least as complicated than its one-dimensional counterpart, a

[1]Axiomatizations for different two-dimensional logics can be found in, for example, Segerberg (1973), Davies and Humberstone (1980), Kaplan (1989), and Fritz (2013).

[2]Hazen (1978) develops a Fitch-style natural deduction system, except that it only contains an actuality operator alongside the modal operators.

[3]First-order axiomatic systems with an actuality operator have been investigated in Hodes (1984) and Stephanou (2005).

[4]Although, as we shall see in due course, only some two-dimensional operators will be said to be explicitly epistemic. In section 5 we shall explore just some of those questions concerning quantification in order to set up a system intended to represent the discourse involving a priori knowledge following Chalmer's version of two-dimensional semantics. In Ray and Lampert (manuscript), we develop a Lemmon-style natural deduction system for quantified two-dimensional modal logic involving actuality and fixedly operators, although two-dimensional developments of systems in the style of Fitch and Gentzen are also yet to be explored.

[5]During the time this paper was being reviewed Gilbert (2016) was published, developing two-dimensional tableaux independently from the present paper. The system considered by Gilbert is basically David and Humberstone's propositional $\mathbf{S5}\mathcal{AF}$ with the addition of a *Ref* operator. In addition, Gilbert presents some decidability results for different two-dimensional logics, which of course will not hold for the first-order systems herein considered. Many thanks to Shawn Standefer and an anonymous reviewer for pointing me to Gilbert's paper.

[6]Not everyone is happy with this. Poggiolesi and Restall (2012) accuse labelled systems of exploiting explicit semantic notions in the proof system, which is unacceptable, so they claim, on the basis that the latter should employ purely syntactic tools. We do not engage in this debate here, although a response against the charge of "semantic pollution" in labelled systems can be found in Read (2015).

simple proof system for it is readily motivated as a useful tool for philosophers. Lastly, on philosophical grounds, the modularity of 2D-tableaux over different notions of validity presents itself as a very attractive property, for it is far from obvious how to achieve the same modularity with axiomatic proof systems.

The paper is divided in the following way: in §2 we provide a quick overview of the rationale behind the introduction of actuality, fixedly, distinguishedly, and *Ref* operators in two-dimensional modal languages; in §3 we present sound and complete semantic tableaux for $\mathbf{S5}_{2D}$ encompassing the operators listed above; in §4 we show how to modulate the 2D-tableaux for different notions of validity, namely, local, general, and diagonal validity; in §5 we show how tableaux for different systems — including, especially, a logic for epistemic two-dimensional semantics — are yielded based on the methods herein presented; finally, in §6 we discuss further the problem of expressive incompleteness in multidimensional languages, which has motivated the addition of some of the operators already mentioned, by focusing on several philosophical consequences and questions raised by introducing the distinguishedly operator in two-dimensional modal languages, some of which bear directly upon how two-dimensional operators, in particular, diagonal necessity operators, should be intuitively understood. Proofs of soundness and completeness for the several systems discussed here can be found in the appendices A, B, and C.

2 Two-Dimensional Modal Logic

2.1 Actually and Fixedly

Crossley and Humberstone defined an **S5** modal logic with an actuality operator, $\mathcal{A}$,[7] whose motivation was the untranslatability in a modal language of sentences such as

(1) It is possible for everything that is actually red to be shiny.

As suggested by Crossley and Humberstone, once we enrich our modal language with $\mathcal{A}$, we can translate (1) as follows:

(2) $\Diamond(\forall x)(\mathcal{A}(red(x)) \supset (shiny(x)))$

The resulting logic is a conservative extension of **S5** called $\mathbf{S5}\mathcal{A}$, where a Kripke model is a triple, $\mathcal{M} = \langle W, w*, V\rangle$, consisting of a non-empty set of 'possible worlds', W, a distinguished member of W, $w*$, called the 'actual world', and a valuation function, V, from propositional variables and members of W to truth-values. The semantics for the modal operators is as usual, and we say that for any sentence, φ, $\mathcal{A}\varphi$ is true at a world just in case φ is true at the actual world, i.e. $\mathcal{M}, w \vDash \mathcal{A}\varphi$ if and only if $\mathcal{M}, w* \vDash \varphi$.

Besides an actuality operator, Crossley and Humberstone also introduced the fixedly operator, $\mathcal{F}$, which works as a universal quantifier over worlds taken as actual. Its original semantics is rather complex. A relation of variance between models is defined such that a model $\mathcal{M}'$ is a variant of $\mathcal{M}$ (written $\mathcal{M}' \approx \mathcal{M}$) if and only if $\mathcal{M}'$ is like $\mathcal{M}$ except possibly

[7] We avoid using quotation marks for the sake of presentation, but use-mention distinctions should be clear given the context.

with respect to which world is actual. Thus, $\mathcal{M}, w \vDash \mathcal{F}\varphi$ if and only if for all $\mathcal{M}' \approx \mathcal{M}$, $\mathcal{M}', w \vDash \varphi$. The introduction of $\mathcal{F}$ was motivated by the fact that $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ comes out as an axiom in **S5**$\mathcal{A}$, despite of its "intuitive invalidity."[8] For the axiom says that whatever is actually true is necessarily actually true, but even though grass is actually green, this is not necessarily so. An alternative to $\Box$ as representing necessity, then, would be the concatenation $\mathcal{FA}$, which seems to deliver a much stronger sense of necessity indeed: if $\mathcal{FA}\varphi$ holds, then φ is true at every world taken as actual. Moreover, $\mathcal{A}\varphi \supset \mathcal{FAA}\varphi$ is not valid in **S5**$\mathcal{AF}$, i.e. **S5**$\mathcal{A}$ with the addition of $\mathcal{F}$.[9] Thus, although whatever is actually true is also $\Box$-necessarily true, it is not $\mathcal{FA}$-necessarily true, in which case we might take $\mathcal{FA}$ as a more faithful representation of the necessity we have in mind when we deny that actual truths are necessary.

Subsequently, Davies and Humberstone offered a doubly-indexed semantics for the logic **S5**$\mathcal{AF}$ with formulas being evaluated with respect to a pair of worlds, whereby it becomes unnecessary to define variance between models for the semantics of the fixedly operator. A new and elegant semantic clause for $\mathcal{F}$ is given as follows: $\mathcal{M}^v_w \vDash \mathcal{F}\varphi$ if and only if for all $z \in W$, $\mathcal{M}^z_w \vDash \varphi$, where the upper world is the actual world under consideration (the first dimension), and the lower world is the world of evaluation (the second dimension).

This doubly-indexed semantics accounts for the two-dimensional flavour of the system since now we can evaluate sentences as being true at a world relative to a certain world considered as actual. A difference in the models for **S5**$\mathcal{AF}$ is that Davies and Humberstone define them as pairs, $\mathcal{M} = \langle W, V \rangle$, rather than the triples with a distinguished element as in **S5**$\mathcal{A}$. The actual world is deposed from the models because in **S5**$\mathcal{AF}$ it is the upper world in the evaluation of formulas that represents which world is actual. The result are 'floating' actual worlds in the same model, rather than a unique and distinguished one. Finally, there is also a need to adjust the semantic clause for $\mathcal{A}$ in **S5**$\mathcal{AF}$: $\mathcal{M}^v_w \vDash \mathcal{A}\varphi$ if and only if $\mathcal{M}^v_v \vDash \varphi$. The actuality operator now has a 'copy down' function: it copies the upper world to the lower one. Hence, rather than pointing at a distinguished element identified in the models — the actually-actual world, as it were — $\mathcal{A}$ points to the upper world, whatever it is, with respect to which the relevant formula is being evaluated.

2.2 Introducing *distinguishedly*

As mentioned above, Davies and Humberstone argue that in **S5**$\mathcal{AF}$ we can represent an interesting and stronger sense of necessity, according to which a sentence can be true at a world *whichever world is taken as actual*, by simply concatenating $\mathcal{F}$ and $\mathcal{A}$. This results in the well-known 'truth on the diagonal' that has been widely used in discussions concerning the a priori.[10] According to Davies and Humberstone (1980, p. 3), $\mathcal{FA}$ provides us with "a formal rendering of a distinction invoked by Gareth Evans between deep and superficial necessity," which was used by Evans to account for cases where a statement was contingently true but knowable a priori — namely, the well-known examples in Kripke (1980). Davies and

[8] Humberstone (2004, p. 21).

[9] We should point out that, originally, Crossley and Humberstone use the formula $\mathcal{A}\varphi \supset \mathcal{FA}\varphi$, which is fine since the repeated actuality operator can be deleted in this case without loss.

[10] In particular, see Stalnaker (1978), Jackson (1998), and Chalmers (1996, 2004) for different usages of this notion.

Humberstone suggested that Evans' distinction could be formally represented in the sense that a sentence, φ, is *superficially necessary* just in case $\Box\varphi$ holds, and *deeply necessary* just in case $\mathcal{FA}\varphi$ holds. The idea is that a sentence would be a priori knowable just in case it is deeply necessary, i.e. if and only if its $\mathcal{FA}$-modalization holds. Thus, assuming that it is a contingent matter which world turns out to be actual, the sentence $\mathcal{A}\varphi \equiv \varphi$, for instance, where φ is any contingent truth, is both contingent and a priori, for its $\mathcal{FA}$-modalization is valid even though its $\Box$-modalization is not.[11] Although there is nothing epistemic built into the formal system since the modalities are alethic, arguably, the resulting logic can be used in this way to represent some important a priori truths, thereby shedding some light on the contingent a priori.

Now, **S5**$\mathcal{AF}$ is very impressive when it comes to its expressiveness. In addition to necessity, possibility, and actuality, we also have deep necessity, which, as alluded above, has been used to represent apriority.[12] Moreover, we can also define the dual of $\mathcal{F}$, which we call "shiftably", by the following semantic clause: $\mathcal{M}^v_w \vDash \mathcal{S}\varphi$ if and only if for some $z \in W$, $\mathcal{M}^z_w \vDash \varphi$.[13] Since a sentence is deeply necessary just in case its $\mathcal{FA}$-modalization holds, we suggest that a sentence is *deeply possible* just in case its $\mathcal{SA}$-modalization holds.[14] However, now it is possible to generate sentences such as the following:

(3) It is deeply possible for everything that is actually red to be shiny.

We should clearly be able to formalize (3) in the logic of deep necessity. But how should we formalize it? As it happens, (3) is a two-dimensional analogue of (1), which motivated the introduction of $\mathcal{A}$ in a basic modal language. And, similarly, if we bring back a distinguished element in the models for **S5**$\mathcal{AF}$, then there will be no way to formalize (3) if we take "actually" as referring to the *actually*-actual world, or the distinguished element of the model, as we take (3) to be suggesting. For instance, the most intuitive strategy to formalize (3) would be to translate "actually" by $\mathcal{A}$, thereby leading to (4):

(4) $\mathcal{SA}(\forall x)(\mathcal{A}(red(x)) \supset (shiny(x)))$

Nevertheless, $\mathcal{A}$ now points to the world introduced by $\mathcal{S}$, and not to the actually-actual world, as we claim it should. The actuality operator is relativized, as it were, in **S5**$\mathcal{AF}$: again, it is just a copy down operator. Since $\mathcal{A}$ becomes sensitive to embedding in $\mathcal{F}$ and $\mathcal{S}$ contexts, this leaves us unable to say in the language things we want to be able to say, such as that something is actually so, and not just actually so relative to a world taken as actual.

In **S5**$\mathcal{A}$ we have the means to refer back to the distinguished world whenever we want, but this is not generally the case for **S5**$\mathcal{AF}$ even if we add a distinguished world to the models. In order for this to be possible we suggest adding — besides a distinguished world to the models — a new operator to the language, called "distinguishedly", which takes any world in the upper position to the distinguished one — hence, distinguishedly works as an $\mathcal{F}$-

[11] Analogously, one could get necessary truths that are only a posteriori knowable, for instance, by substituting any empirical truth for φ in $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$.

[12] In Restall (2012) and Fritz (2013), a priori operators behave exactly as $\mathcal{FA}$. Thus, in §5 we develop tableau systems taking $\mathcal{FA}$ as a single operator.

[13] The shiftably operator was previously defined in Ray and Lampert (manuscript).

[14] Hanson (2006, p. 452), defends that, to a certain extent, deep necessity and possibility seem to capture our intuitions about necessity and possibility in general even better than superficial necessity and possibility.

or $\mathcal{S}$-inhibitor.[15] Finally, if we denote distinguishedly by $\odot$, and let $w*$ be the distinguished element of the models, we can present its semantics as follows:

$$\mathcal{M}_w^v \vDash \odot\varphi \text{ if and only if } \mathcal{M}_w^{w*} \vDash \varphi$$

The idea is that we can now translate (3) into our language as the following:

(5) $\mathcal{SA}(\forall x)(\odot\mathcal{A}(red(x)) \supset (shiny(x)))$

Thus we have both a way of denoting any floating actual world by using $\mathcal{F}$, $\mathcal{S}$, and $\mathcal{A}$, as well as the distinguished element in the models by using $\odot$. For obvious reasons, we sometimes refer to the latter as a *rigid* actuality operator, thereby leaving to $\mathcal{A}$ the title of *non-rigid* actuality operator. There are several philosophical issues to be discussed concerning these, and we address some of them in §6.

2.3 *Ref*

It would be interesting enough to have a proof system for all the operators discussed above. However, we can increase the expressive power of our two-dimensional logic even more if we add what Cresswell (1990) calls the "*Ref*" operator. This is very similar to Vlach's (1973) "then" operator for temporal logics, which is usually studied in the company of Prior's (1968) "now" operator, the tense analogue of $\mathcal{A}$.[16] The need for such an operator comes from an attempt to formalize sentences such as

(6) It might have been that, if everyone then rich might have been poor, then someone is happy.

What this example is purported to show is the following.[17] The term "then", similarly to "actual", points to a certain world. Except that where "actual" points to the distinguished element in the models (or the upper world), "then" points to the possible world introduced in the beginning of (6). Thus, we cannot formalize (6) as

(7) $\Diamond(\Diamond(\forall x)(\mathcal{A}(rich(x)) \supset (poor(x))) \supset (\exists x)(happy(x)))$

What (7) says, in effect, is that there is a possible world, w, such that, for some possible world z, if everyone who is rich at the actual world, $w*$, is poor at z, then someone is happy at w.[18] However, what we want is to quantify over whoever is rich at w, and not $w*$. In order to do that we need to somehow mark the first possible world introduced, w, such that w will now be taken as the reference world. We do this by using *Ref*, which we symbolize henceforth as $\otimes$:[19]

[15] In Humberstone (1982, p. 104), $\mathcal{A}$ is called an inhibitor (in a logic without $\mathcal{F}$), for it "protects what is in its scope from the influence of an outlying modal operator."

[16] This operator can also be found in Stalnaker (1978), being symbolized by †. On p. 320, Stalnaker offers $\Box$† as an apriority operator, which is equivalent to $\mathcal{FA}$. More on this in §4.1.

[17] This example can be found in Sider (2010, p. 225).

[18] Assuming, of course, a rigid actuality operator, where $\mathcal{A}$ always takes us to the actual world. In a two-dimensional logic, $\mathcal{A}$ copies down the upper world, whatever it is, whereby it is enough for the purposes of (7) that the upper world is not w.

[19] Sider formalizes *Ref* by using $\times$. But since we use $\times$ as a syntactic mark for when a branch of tableau closes, we choose a slightly different symbol instead.

$\mathcal{M}_w^v \vDash \otimes\varphi$ if and only if $\mathcal{M}_w^w \vDash \varphi$

Now we can formalize (6) as follows:

(8) $\Diamond \otimes (\Diamond(\forall x)(\mathcal{A}(rich(x)) \supset (poor(x))) \supset (\exists x)(happy(x)))$

It can be easily checked that (8) gives us the correct formal rendering of (6). Just as $\mathcal{A}$ is a copy down operator, $\otimes$ can be seen as a copy up operator, for it copies up the world of evaluation to the upper position, the actual world under consideration. In the next section we present tableau methods for first-order $\mathbf{S5}_{2D}$, consisting of the two-dimensional operators alluded thus far alongside the modal operators for necessity and possibility.

3 Tableaux for $\mathbf{S5}_{2D}$

3.1 Syntax and Semantics for $\mathbf{S5}_{2D}$

Definition 3.1 (First-order language) For the language $\mathcal{L}_{2D}$, let $\{c_1, c_2, ...\}$ be a set of *constant symbols*, $\{x_1, x_2, ...\}$ a set of *individual variables*, and $\{P_1^n, P_2^n, ...\}$ a set of n-place *predicate symbols* for each $n \in \mathbb{N}$. The terms t and formulas φ are recursively generated by the following grammar ($i, n \in \mathbb{N}$):

$$t ::= c_i \mid x_i$$

$$\varphi ::= P_i^n(t_1, ..., t_n) \mid t = t' \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathcal{A}\varphi \mid \odot\varphi \mid \otimes\varphi \mid \mathcal{F}\varphi \mid \exists x_i\varphi$$

The other Boolean connectives and the universal quantifier $\forall x_i\varphi$ are defined as usual. Moreover, we define $\Diamond$ as $\neg\Box\neg$, and $\mathcal{S}$ as $\neg\mathcal{F}\neg$.[20] In addition, we define the set of *basic formulas* of $\mathcal{L}_{2D}$. Where AT is the set of atomic formulas of $\mathcal{L}_{2D}$, the basic formulas, ψ, of $\mathcal{L}_{2D}$, are generated as follows:

$$\psi ::= AT \mid \neg\psi$$

With respect to the models, we have already mentioned different ways to define them for the two-dimensional case. We can either define *centered* models, with a distinguished element, whereby $\mathcal{F}$ changes the models to variant ones, or we can evaluate formulas with respect to pairs of worlds with $\mathcal{F}$ quantifying universally over the first coordinate of every pair. In the latter case, there was originally no need for a distinguished point in the models, since the first coordinate of every pair plays the role of the actual world under consideration anyway. But since we have a distinguishedly operator in our language, we definitely want a distinguished element fixed in the models. Our strategy, then, will be to evaluate formulas with respect to pairs of worlds, since this will provide us with a more uniform treatment for different two-dimensional systems, in which case the set of possible worlds in the models is defined accordingly in order to contain ordered pairs rather than single worlds. This can be accomplished by simply taking the usual set W of possible worlds and letting it be $W \times W$, with $\langle w*, w*\rangle$ being its distinguished element.[21] Consequently, $\odot\mathcal{A}$ will have the function of

[20] Also, in what follows we use $a, b, c, ...$ for constant symbols and $x, y, z, ...$ for individual variables.

[21] Since $w* \in W$ is the distinguished element of W. More on this below.

pointing to the distinguished element of two-dimensional models. We also define accessibility relations for both $\Box$ and $\mathcal{F}$ formulas, which we take to be equivalence relations, but which might be restricted to different properties as usual, generating two-dimensional versions of different modal logics like **T**, **B**, and so on.

Definition 3.2 (Constant domain 2D-centered models) A *constant domain 2D-centered model* is a tuple, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, such that

- $W = Z \times Z$ for some set Z,
- $\langle w*, w*\rangle$ is a distinguished element of W,[22]
- $\mathcal{R}_\Box \subseteq W \times W$, the $\Box$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle$,
- $\mathcal{R}_\mathcal{F} \subseteq W \times W$, the $\mathcal{F}$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle$,
- $\mathscr{D}$ is a non-empty domain of quantification, and
- V is a function assigning to each constant c_i of $\mathcal{L}$ and $\langle v, w\rangle \in W$, an object $V(c_i, \langle v, w\rangle) \in \mathscr{D}$, and to each n-place predicate symbol P^n_i and $\langle v, w\rangle \in W$, a set $V(P^n_i, \langle v, w\rangle) \subseteq \mathscr{D}^n$.

For the moment we want our constants to be *rigid designators*, that is, with a world-invariant designation, not only with respect to $\mathcal{R}_\Box$, but also with respect to $\mathcal{R}_\mathcal{F}$-accessible worlds. Because we have two indices to keep track of in our semantics, there are two ways constants can be said to denote rigidly: a constant is $\mathcal{R}_\Box$-*rigid* if it denotes the same objects with respect to $\mathcal{R}_\Box$-accessible pairs of worlds, and a constant is $\mathcal{R}_\mathcal{F}$-rigid if it denotes the same objects with respect to $\mathcal{R}_\mathcal{F}$-accessible pairs of worlds. Since we want constants to be rigid in both senses at least for now, we assume the following constraints on models:[23] where c_i is any constant symbol in $\mathcal{L}_{2D}$,

($\mathcal{R}_\Box$-rigidity) For every $v, w, z \in Z$, if $\langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle$, then $V(c_i, \langle v, w\rangle) = V(c_i, \langle v, z\rangle)$,

($\mathcal{R}_\mathcal{F}$-rigidity) For every $v, w, z \in Z$, if $\langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle$, then $V(c_i, \langle v, w\rangle) = V(c_i, \langle z, w\rangle)$.

[22] Thus, by following the common practice of singling out an element of the set of worlds in the models as distinguished, we get an ordered pair, rather than the usual single distinguished world $w*$. But this in turn might give one reason to demur: the distinguished element of the models is more than often interpreted in an intuitive way as the actual world, but certainly the pair $\langle w*, w*\rangle$ should not be understood as *the* actual world! We agree. An alternative would be to define (constant domain) 2D-centered models as $\mathcal{M} = \langle W, w*, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, where $W = Z \times Z$ and $w* \in Z$. The reason we take the distinguished element to consist of an ordered pair is because it provides us with a more uniform account of truth in a model. Since every formula is being evaluated with respect to pairs of worlds, truth in a model must, too, be relativized to a pair. And since truth in a model is defined as truth at a distinguished point, a distinguished pair is called for. (Thus, we say that truth at $\langle w*, w*\rangle$ can be intuitively understood as truth at the actual world with respect to itself.) If one dislikes this option we offer the alternative definition in its place. This has no direct bearing on the tableaux

[23] Those are almost the same as the ones defined by Holliday and Perry (2014). In §5 we define a different system with non-rigid constants for a new kind of accessibility relation.

Such conditions, of course, could be lifted in favour of non-rigid constants. But, for simplicity, we start with rigid constants, moving to non-rigid designation in §5. We also impose a third condition on models corresponding to an important feature of the semantics presented by Davies and Humberstone:

(Neutrality) If φ is a basic formula, then for every $v, w, z \in Z$, $\mathcal{M}^v_w \models \varphi$ if and only if $\mathcal{M}^z_w \models \varphi$.

This constraint is intended to make the truth of basic formulas sensitive only to the second coordinate in a pair, in which case the actual world under consideration turns out to be a free parameter for atoms (and their negations) of the language. With this assumption at hand, our models will validate $\mathcal{F}\varphi \equiv \varphi$ for every basic formula φ, corresponding thereby to Davies and Humberstone's models.[24]

Definition 3.3 (Truth) We define 'φ is true at w relative to v in $\mathcal{M}$', written $\mathcal{M}^v_w \models \varphi$, by recursion on φ. For a pair $\langle v, w\rangle \in W$, and a valuation V in $\mathcal{M}$,

$$
\begin{array}{lcl}
\mathcal{M}^v_w \models P^n_i(t_1, ..., t_n) & \iff & \langle V(t_1, \langle v, w\rangle), ..., V(t_n, \langle v, w\rangle)\rangle \in V(P^n_i, \langle v, w\rangle); \\
\mathcal{M}^v_w \models t = t' & \iff & V(t, \langle v, w\rangle) = V(t', \langle v, w\rangle); \\
\mathcal{M}^v_w \models \neg\varphi & \iff & \mathcal{M}^v_w \not\models \varphi \\
\mathcal{M}^v_w \models \varphi \wedge \psi & \iff & \mathcal{M}^v_w \models \varphi \text{ and } \mathcal{M}^v_w \models \psi; \\
\mathcal{M}^v_w \models \Box\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle, \text{ then } \mathcal{M}^v_z \models \varphi; \\
\mathcal{M}^v_w \models \Diamond\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\Box \langle v, z\rangle \text{ and } \mathcal{M}^v_z \models \varphi; \\
\mathcal{M}^v_w \models \mathcal{A}\varphi & \iff & \mathcal{M}^v_v \models \varphi; \\
\mathcal{M}^v_w \models \odot\varphi & \iff & \mathcal{M}^{w*}_w \models \varphi; \\
\mathcal{M}^v_w \models \otimes\varphi & \iff & \mathcal{M}^w_w \models \varphi; \\
\mathcal{M}^v_w \models \mathcal{F}\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle, \text{ then } \mathcal{M}^z_w \models \varphi; \\
\mathcal{M}^v_w \models \mathcal{S}\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle \mathcal{R}_\mathcal{F} \langle z, w\rangle \text{ and } \mathcal{M}^z_w \models \varphi; \\
\mathcal{M}^v_w \models \forall x_i\varphi & \iff & \text{for every } x_i\text{-variant } V' \text{ of } V, \mathcal{M}^v_w \models \varphi[c_i/x_i]; \\
\mathcal{M}^v_w \models \exists x_i\varphi & \iff & \text{for some } x_i\text{-variant } V' \text{ of } V, \mathcal{M}^v_w \models \varphi[c_i/x_i];
\end{array}
$$

For V' to be an x-variant of V is just for V' and V to disagree at most on x. Lastly, a sentence φ is *false* at w relative to v in $\mathcal{M}$ if and only if it is not true at w relative to v in $\mathcal{M}$.

Definition 3.4 (Logical properties) A sentence, φ, is *true simpliciter* under $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if and only if φ is true at $w*$ relative to $w*$ in $\mathcal{M}$ (i.e. if and only if $\mathcal{M}^{w*}_{w*} \models \varphi$); a sentence φ is *valid* if and only if it is true *simpliciter* under every $\mathcal{M}$; and a sentence φ is a *logical consequence* of a set of sentences Γ if and only if for every $\mathcal{M}$, if $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$.

Notice that we depart from Crossley and Humberstone's notion of general validity — truth at all worlds in all models — by adopting what they call 'real-world validity', which is just the definition of validity given in Kripke (1963, p. 69). In §4, however, we show how to adapt the tableaux for general validity as well.

[24]In §5 we define another system in which this constraint is abandoned, and we discuss some philosophical issues related to it.

3.2 2D-Tableaux

Tableaux for two-dimensional modal logic are similar to the modal tableaux with numeric indices presented in both Fitting and Mendelsohn (1998) and Priest (2008). In both of these cases, every node of a tableau is an indexed formula, where the index is a numeral denoting (intuitively) a possible world according to which the relevant formula is evaluated.[25] In the 2D case, however, we need double indexing in order to denote (intuitively) the actual world under consideration — the upper index — as well as the usual possible world — the lower index. Let n,m,... denote lower indices, and i,j,... denote upper indices.[26] For the numeric index '0', we fix its interpretation to name $w*$, in which case the pair of indices $\langle 0, 0\rangle$ denotes the distinguished element in the respective model. Moreover, a *doubly-indexed formula* is an expression, $[\varphi]^i_n$, where i and n constitute a pair of indices, $\langle i, n\rangle$, and φ is a formula. All doubly-indexed formulas are enclosed within brackets.

The root of a 2D-tableau always contains the negation of the formula we are attempting to prove doubly-indexed by the pair $\langle 0, 0\rangle$. So, for any formula, φ, a 2D-tableau proof of φ begins with $[\neg\varphi]^0_0$. This says, intuitively, that $\neg\varphi$ is true at the possible world denoted by '0' relative to '0' taken as actual. On the other hand, if we attempt to prove φ from a non-empty set of premises, we add $[\psi]^0_0$, for every premise ψ, to the first lines of the 2D-tableau.

Now we present the rules for 2D-tableaux. The rules for the Boolean connectives are just the ones found in Smullyan (1995) grouped into conjunctive and disjunctive rules, except that they now receive double indexing. Moreover, for the Boolean connectives, the double indexing of a formula is preserved to its immediate descendants.

Definition 3.5 (Conjunctive Rules)

For any pair of indices $\langle i, n\rangle$,

$$\begin{array}{cccc} [\varphi \wedge \psi]^i_n & [\neg(\varphi \vee \psi)]^i_n & [\neg(\varphi \supset \psi)]^i_n & [\varphi \equiv \psi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_n & [\neg\varphi]^i_n & [\varphi]^i_n & [\varphi \supset \psi]^i_n \\ [\psi]^i_n & [\neg\psi]^i_n & [\neg\psi]^i_n & [\psi \supset \varphi]^i_n \end{array}$$

Definition 3.6 (Disjunctive Rules)

For any pair of indices $\langle i, n\rangle$,

$$\begin{array}{cccc} [\varphi \vee \psi]^i_n & [\neg(\varphi \wedge \psi)]^i_n & [\varphi \supset \psi]^i_n & [\neg(\varphi \equiv \psi)]^i_n \\ \swarrow \quad \searrow & \swarrow \quad \searrow & \swarrow \quad \searrow & \swarrow \quad \searrow \\ [\psi]^i_n \quad [\varphi]^i_n & [\neg\varphi]^i_n \quad [\neg\psi]^i_n & [\neg\varphi]^i_n \quad [\psi]^i_n & [\neg(\varphi \supset \psi)]^i_n \quad [\neg(\psi \supset \varphi)]^i_n \end{array}$$

Moreover, we define a double negation rule as well:

[25] In Priest's (2008) case, some nodes may be expressions such as 'nrm', where n and m are numerals such that the possible world denoted by m is r-accessible from n. Fitting and Mendelsohn (1998), on the other hand, do not make use of single nodes reflecting the accessibility relations of the system in question. Rather, formulas may be prefixed by sequences of indices such as $\sigma.n$, which means intuitively that n is a world accessible from σ; whereby there is no need to add nodes with the single purpose of denoting an accessibility relation.

[26] We avoid using the letter 'o' to denote lower indices since it may be confused with the number 0.

Definition 3.7 (Double Negation Rule)

For any pair of indices $\langle i, n \rangle$,

$$\begin{array}{c} [\neg\neg\varphi]^i_n \\ \downarrow \\ [\varphi]^i_n \end{array}$$

Next we present the rules for modal and two-dimensional operators. These are grouped as follows. The actuality rules contain the rules for $\mathcal{A}$, $\odot$, and $\otimes$; the possibility rules contain the rules for $\Diamond$ and $\mathcal{S}$; and the necessity rules comprise the rules for $\Box$ and $\mathcal{F}$. For the possibility rules there is a constraint on the indices added to the branch where the rule is applied, namely, they have to be new to the branch. This is just the usual set up for possibility rules in modal tableaux.[27] But since the formulas in 2D-tableaux are doubly-indexed, this ought to be generalized for two indices. Next we state the rules for modal and two-dimensional operators:

Definition 3.8 (Actuality Rules)

For any pair of indices $\langle i, n \rangle$,

$$\begin{array}{cccccc} [\mathcal{A}\varphi]^i_n & [\neg\mathcal{A}\varphi]^i_n & [\odot\varphi]^i_n & [\neg\odot\varphi]^i_n & [\otimes\varphi]^i_n & [\neg\otimes\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_i & [\neg\varphi]^i_i & [\varphi]^0_n & [\neg\varphi]^0_n & [\varphi]^n_n & [\neg\varphi]^n_n \end{array}$$

Definition 3.9 (Possibility Rules)

If the index z is new to the branch,

$$\begin{array}{cccc} [\Diamond\varphi]^i_n & [\neg\Box\varphi]^i_n & [\mathcal{S}\varphi]^i_n & [\neg\mathcal{F}\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_z & [\neg\varphi]^i_z & [\varphi]^z_n & [\neg\varphi]^z_n \end{array}$$

Definition 3.10 (Necessity Rules)

For every index z occurring on the branch,

$$\begin{array}{cccc} [\Box\varphi]^i_n & [\neg\Diamond\varphi]^i_n & [\mathcal{F}\varphi]^i_n & [\neg\mathcal{S}\varphi]^i_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ [\varphi]^i_z & [\neg\varphi]^i_z & [\varphi]^z_n & [\neg\varphi]^z_n \end{array}$$

In addition, we define the following rule corresponding to (Neutrality):

Definition 3.11 (Upper exchange Rule)

If φ is a basic formula, then for any index j on the branch,

$$\begin{array}{c} [\varphi]^i_n \\ \downarrow \\ [\varphi]^j_n \end{array}$$

[27] See, for instance, Fitting and Mendelsohn (1998, p. 49).

Finally, we present the rules for identity and quantifiers. The rules for the quantifiers are just the usual ones found in tableaux for constant domain modal logic, so there is nothing particularly new to them.[28] We adopt constant domain here because of its convenience, but it should be clear how the rules for variable domains can be similarly adapted. In the case of identity, one of the rules allows us some freedom with respect to the lower index, just as in the basic modal case for identity, while keeping the upper index unmoved.

Definition 3.12 (Identity Rules)

For any constant c and pair of indices $\langle i, n\rangle$ already occurring on the branch,

$$[c = c]^i_n$$

If $[c = d]^i_n$ and $[\varphi(c)]^i_m$ already occur on the branch,

$$\begin{array}{c}[c = d]^i_n \\ [\varphi(c)]^i_m \\ \downarrow \\ [\varphi(d)]^i_m\end{array}$$

Definition 3.13 (Universal Rules)

For any constant c and pair of indices $\langle i, n\rangle$,

$$\begin{array}{cc}[(\forall x)\varphi]^i_n & [\neg(\exists x)\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi[c/x]]^i_n & [\neg\varphi[c/x]]^i_n\end{array}$$

Definition 3.14 (Existential Rules)

For any constant c not occurring on the branch, and pair of indices $\langle i, n\rangle$,

$$\begin{array}{cc}[(\exists x)\varphi]^i_n & [\neg(\forall x)\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi[c/x]]^i_n & [\neg\varphi[c/x]]^i_n\end{array}$$

We say that a branch $\mathfrak{b}$ of a 2D-tableau is *closed* just in case for some formula, φ, and a pair of indices $\langle i, n\rangle$, both $[\varphi]^i_n$ and $[\neg\varphi]^i_n$ occur on $\mathfrak{b}$. Otherwise, we say the 2D-tableau is *open*. A 2D-tableau is *closed* just in case all of its branches are closed. Finally, we define what it is to be a 2D-tableau proof:

Definition 3.15 (2D-tableau proof) A *2D-tableau proof* of a sentence, φ, is a closed 2D-tableau for $[\neg\varphi]^0_0$. Moreover, for a (possibly empty) set of sentences, Γ, and a sentence, φ, if φ is provable from Γ in $\mathbf{S5}_{2D}$ we write $\Gamma \vdash \varphi$. As usual, φ is a *theorem* in $\mathbf{S5}_{2D}$ just in case it is provable from an empty set of premises.

Soundness and completeness proofs for $\mathbf{S5}_{2D}$ can be found in appendix A. It is illustrative to see how 2D-tableaux work, so in what follows we prove some theorems of $\mathbf{S5}_{2D}$. We begin with the propositional portion of $\mathbf{S5}_{2D}$, and then we exhibit a quantified case:

[28] See, for instance, Priest (2008, p. 310).

$\vdash \mathcal{F}\mathcal{A}p \equiv \Box p$

$$1.\ [\neg(\mathcal{F}\mathcal{A}p \equiv \Box p)]^0_0$$

2. $[\neg(\mathcal{F}\mathcal{A}p \supset \Box p)]^0_0$	3. $[\neg(\Box p \supset \mathcal{F}\mathcal{A}p)]^0_0$
4. $[\mathcal{F}\mathcal{A}p]^0_0$	12. $[\Box p]^0_0$
5. $[\neg\Box p]^0_0$	13. $[\neg\mathcal{F}\mathcal{A}p]^0_0$
6. $[\neg p]^0_1$	14. $[\neg\mathcal{A}p]^1_0$
7. $[\mathcal{A}p]^0_0$	15. $[\neg p]^1_1$
8. $[\mathcal{A}p]^1_0$	16. $[p]^0_0$
9. $[p]^0_0$	17. $[p]^0_1$
10. $[p]^1_1$	18. $[p]^1_1$
11. $[p]^0_1$	$\times$
$\times$	

Items 2, 3, 4, 5, 12, and 13 result from truth-functional rules, while 6, 16, and 17 result from the possibility and necessity rules for $\neg\Box$ and $\Box$, respectively. 7, 8, and 14 result from the necessity and possibility rules for $\mathcal{F}$ and $\neg\mathcal{F}$, respectively, while 9, 10, and 15 are just applications of the rules for $\mathcal{A}$. Finally, the upper exchange rule was used in 11 and 18. Another example, this time involving a failed tableau proof attempt, is the following:

$\nvdash \mathcal{S}\Diamond p \supset \Box\mathcal{F}\Box p$

$$\begin{array}{c}
1.\ [\neg(\mathcal{S}\Diamond p \supset \Box\mathcal{F}\Box p)]^0_0 \\
2.\ [\mathcal{S}\Diamond p]^0_0 \\
3.\ [\neg\Box\mathcal{F}\Box p]^0_0 \\
4.\ [\neg\mathcal{F}\Box p]^0_1 \\
5.\ [\neg\Box p]^2_1 \\
6.\ [\neg p]^2_3 \\
7.\ [\Diamond p]^4_0 \\
8.\ [p]^4_5
\end{array}$$

The only branch of the tableau remains open. Since this case only involves the propositional portion of $\mathbf{S5}_{2D}$, a propositional model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{D}, V\rangle$, can be constructed from the open branch above in the following way. Where k is any single index, $\mathfrak{b}$ is the open branch above, and $Z = \{k \mid k \in \mathfrak{b}\}$, set $W = Z \times Z$, where $\langle 0, 0\rangle$ is a distinguished element of W. Moreover, for every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle\mathcal{R}_\Box\langle i, m\rangle$; similarly, for every $\langle i, n\rangle, \langle j, n\rangle \in W$, set $\langle i, n\rangle\mathcal{R}_\mathcal{F}\langle j, n\rangle$. Since we are assuming that both accessibility relations are equivalence relations, we drop any mention of them in what follows. Finally, for any propositional letter, say, q, if $[q]^i_n$ occurs on $\mathfrak{b}$, then $\mathcal{M}^i_n \vDash q$; if, on the other hand, $[\neg q]^i_n$ occurs on $\mathfrak{b}$, then $\mathcal{M}^i_n \nvDash q$; if neither, set either $\mathcal{M}^i_n \vDash q$ or $\mathcal{M}^i_n \nvDash q$. Thus, a countermodel for the open branch in the tableau above will have $Z = \{0, 1, 2, 3, 4, 5\}$, $W = Z \times Z$, $\mathcal{M}^2_3 \nvDash p$, and $\mathcal{M}^4_5 \vDash p$.

The set W contains a long list of pairs generated from Z, but we can represent a countermodel using a 2D-matrix where the vertical axis designates reference or actual worlds and the horizontal axis designates possible worlds:

$$\begin{pmatrix} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & \times & \times & \times & \times & \times & \times \\ 1 & \times & \times & \times & \times & \times & \times \\ 2 & \times & \times & \times & \neg p & \times & \times \\ 3 & \times & \times & \times & \times & \times & \times \\ 4 & \times & \times & \times & \times & \times & p \\ 5 & \times & \times & \times & \times & \times & \times \end{pmatrix}$$

Since $\mathcal{M}_5^4 \vDash p$, it follows that $\mathcal{M}_0^4 \vDash \Diamond p$, and hence $\mathcal{M}_0^0 \vDash \mathcal{S}\Diamond p$. On the other hand, $\mathcal{M}_3^2 \nvDash p$, whence $\mathcal{M}_1^2 \nvDash \Box p$ and $\mathcal{M}_1^0 \nvDash \mathcal{F}\Box p$. Therefore, $\mathcal{M}_0^0 \nvDash \Box\mathcal{F}\Box p$. Finally, in what follows we show how a countermodel can be read-off from a failed proof with quantifiers:

$\nvDash (\exists x)Px \supset (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px$

1. $[\neg((\exists x)Px \supset (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px)]_0^0$
2. $[(\exists x)Px]_0^0$
3. $[\neg(\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px]_0^0$
4. $[Pa]_0^0$
5. $[\neg\mathcal{F}\mathcal{A}\Diamond \otimes Pb]_0^0$
6. $[\neg\mathcal{A}\Diamond \otimes Pb]_0^1$
7. $[\neg\Diamond \otimes Pb]_1^1$
8. $[\neg \otimes Pb]_0^1$
9. $[\neg \otimes Pb]_1^1$
10. $[\neg Pb]_0^0$
11. $[\neg Pb]_1^1$

Again, set $W = Z \times Z$, where Z is the set containing every (single) index occurring on the open branch above, $\mathfrak{b}$, and let $\langle 0, 0\rangle$ be a distinguished element of W. Now we define C as the set of all constants, c, occurring on $\mathfrak{b}$, and $\mathscr{D} = \{c \mid c \in C\}$. Moreover, set $V(c, \langle i, n\rangle) = c$ to each $c \in C$ and $\langle i, n\rangle \in W$, and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n\rangle) = \{\langle c_1, ..., c_n\rangle \mid R(c_1, ..., c_n)_n^i$ occurs on $\mathfrak{b}\}$. Given both rigidity conditions, for any $\langle i, n\rangle, \langle j, m\rangle \in W$ and $c \in C$, set $V(c, \langle i, n\rangle) = V(c, \langle j, m\rangle)$. The following is a countermodel for the open branch above: $Z = \{0, 1\}$, in which case we have $W = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle\}$, $V(P, \langle 0, 0\rangle) = \{a\}$. We can represent the model by the following 2D-matrix:

$$\begin{pmatrix} & 0 & 1 \\ 0 & Pa & \times \\ 1 & \times & \times \end{pmatrix}$$

We prove that $\mathcal{M}_0^0 \not\models (\exists x)Px \supset (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px$. Let V' be the x-variant of V such that $\mathcal{M}_0^0 \models Pa$. Thus, $\mathcal{M}_0^0 \models (\exists x)Px$. However, for some x-variant V' of V, $\mathcal{M}_0^0 \not\models Pb$ and $\mathcal{M}_1^1 \not\models Pb$, in which case we have both $\mathcal{M}_0^1 \not\models \otimes Pb$ and $\mathcal{M}_1^1 \not\models \otimes Pb$. So, $\mathcal{M}_1^1 \not\models \Diamond \otimes Pb$ and $\mathcal{M}_0^1 \not\models \mathcal{A}\Diamond \otimes Pb$, whence $\mathcal{M}_0^0 \not\models \mathcal{F}\mathcal{A}\Diamond \otimes Pb$. Therefore, $\mathcal{M}_0^0 \not\models (\forall x)\mathcal{F}\mathcal{A}\Diamond \otimes Px$.

4 Local, General, and Diagonal Validity

In the semantics for $\mathbf{S5}_{2D}$ we defined validity as truth at the distinguished element of every model. This is just what you would expect as an extension of the canonical notion of validity found in Kripke's early work in modal semantics.[29] However, this has not become the standard definition of validity for modal logics since, after Kripke's early papers, validity is usually defined as *truth at all worlds in every model.* In Crossley and Humberstone (1977) those are called *real world* and *general* validity, respectively.[30] A small terminological difference is that here we use *local* validity for the former.

In basic modal logics without actuality operators both notions of validity are equivalent, but this does not hold in general for languages containing $\mathcal{A}$. For example, the sentence $\mathcal{A}p \supset p$ is locally but not generally valid. This is easy to check, for in a basic modal semantics we can just set $w* \in V(p)$ and $w \notin V(p)$, where w is any world other than $w*$, which makes $\mathcal{A}p$ true at w and p false at w.[31] Since a sentence is generally valid just in case it holds at every world of every model, the above suffices for a model falsifying $\mathcal{A}p \supset p$. On the other hand, its $\mathcal{A}$-modalization, $\mathcal{A}(\mathcal{A}p \supset p)$, is in effect generally valid. This reflects the fact that for any locally valid sentence, ψ, its $\mathcal{A}$-modalization, $\mathcal{A}\psi$, is generally valid.

The set of valid sentences in modal logics containing an actuality operator is conditional on our choice between local and general validity, which brings about important philosophical consequences for the resulting logic. Zalta (1988), for instance, argues that local validity is the best alternative for modal logics since it is defined in terms of truth *simpliciter*, or truth in a model. General validity, on the other hand, bypasses the notion of truth in a model, being thus defined in terms of truth at all worlds. Hence, while local validity clearly characterizes the canonical Tarskian notion of logical truth, general validity does not. Besides, truth at all worlds is best seen as characterizing necessity, and there seems to be no principled reason to identify validity with necessary truth, at least not without an argument.[32] The upshot, according to Zalta, is that modal logics with an actuality operator, when characterized by the right notion of validity, generate contingent logical truths, whose paradigmatic example is $\mathcal{A}p \supset p$. It is straightforward, moreover, to see that this move entails a failure of the rule of necessitation, for even though $\mathcal{A}p \supset p$ is locally valid, its necessitation, $\Box(\mathcal{A}p \supset p)$, is not.[33] Therefore, in logics containing an actuality operator, the rule of necessitation must be restricted accordingly.

[29]See Kripke (1958) and (1963).

[30]Zalta (1988) provides a list with many influential textbooks in modal logic in which the authors define validity as general rather than real-world. More recently, we could also add Fitting and Mendelsohn (1998) and Priest(2008) to the list.

[31]In our case, this has to be adjusted to pairs of worlds.

[32]For more on this, see Nelson and Zalta (2012), and Hanson (2006, 2014) for a response.

[33]This causes the rule of necessitation to fail in $\mathbf{S5}_{2D}$ as well, unless one defines general rather than local validity. In that case, $\mathcal{A}p \supset p$ would not count as a theorem of $\mathbf{S5}_{2D}$.

4.1 Modular tableaux

There are several questions to be asked concerning the best notion of validity in modal logics with actuality operators. Should we count, after all, $\mathcal{A}p \supset p$ as a logical truth? Do we need to have an account of validity directly in terms of truth in a model? Perhaps modal semantics requires logical truth to be delineated differently. Perhaps it does not. Be that as it may, our present task does not involve defending a particular notion of validity, for it is a clear advantage of 2D-tableaux that they afford us with a simple way to modulate between local and general validity, which is a palliative for those uncomfortable with contingent logical truths.[34] With respect to local validity, 2D-tableaux are exactly as they were set up above, that is, the root of a 2D-tableau is the negation of the formula we want to prove, doubly-indexed by $\langle 0, 0\rangle$, i.e. $[\neg\varphi]^0_0$. (We remind the reader that the index '0' is fixed to denote $w*$). Thus, given the soundness and completeness theorems in appendix A, a formula, φ, is locally valid just in case there is a proof of φ (from zero premises) indexed by $\langle 0, 0\rangle$.

In order to make a distinction between tableaux for different kinds of validity, we call tableaux for local validity *local 2D-tableaux.* Next we show how to set up *general 2D-tableaux* corresponding to general validity.

4.1.1 Tableaux for general validity

In the case of general validity, the tableaux are set up as follows:

Definition 4.1 (General 2D-tableaux) A *general 2D-tableau* is defined just like local 2D-tableaux, except that the root of a general 2D-tableau is the negation of the formula we want to prove, doubly-indexed by $\langle n, m\rangle$, i.e. $[\neg\varphi]^n_m$, where n and m denote any indices different from '0', and $n \neq m$.

Thus, if φ is provable from zero premises where its negation was doubly-indexed by $\langle n, m\rangle$ on the root of the tableau, the soundness theorem in the appendix guarantees that for every 2D-centered model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$, φ is true at $\langle n, m\rangle$ under $\mathcal{M}$. Since there are no constraints imposed on neither n nor m besides their being different and distinct from '0', they can pick out any two worlds in any model,[35] whence φ will be generally valid. Now the definition of a 2D-tableau proof ought to be restated, too, for general validity:

Definition 4.2 A *general 2D-tableau proof* of a sentence, φ, is a closed general 2D-tableau for $[\neg\varphi]^n_m$, where n and m denote any indices different from '0', and $n \neq m$.

[34] This was previously shown in Ray and Lampert (manuscript) for natural deduction systems with numeric indices. Since the tableaux presented here work in a similar way, this result could be easily adapted. I am grateful to Greg Ray for helpful suggestions concerning this section.

[35] For languages lacking a distinguishedly operator, one of the indices does not need to be different from '0'. Thus, general 2D-tableaux for $\odot$-free languages might begin with the pair $\langle 0, n\rangle$, where $0 \neq n$. However, things are trickier in languages containing $\odot$. For example, by starting a general 2D-tableau with $\langle 0, 1\rangle$ we would be able to prove $\Box p \supset \odot\mathcal{A}p$, which is valid when (Neutrality) is assumed, but not otherwise. Since in §5 we investigate two-dimensional logics where the models are not constrained by (Neutrality), had we defined general 2D-tableaux without requiring of both indices in the tableau's root that they must be different from '0', we would have to change our definition and incorporate this restriction. Because of this, the requirement that both indices should be different from '0' gives us a more general treatment to deal with general 2D-tableaux.

Furthermore, the generalized notion of consequence corresponding to general validity says that a sentence, φ, is a *general consequence* of a set of sentences Γ if and only if for every model $\mathcal{M}$ and every pair $\langle v, w\rangle \in W$, if $\mathcal{M}^v_w \vDash \gamma$, for all $\gamma \in \Gamma$, then $\mathcal{M}^v_w \vDash \varphi$.

Now we give an example of a general 2D-tableau proof attempt of a sentence that is locally but not generally valid. We use the indices '1','2' for n and m, respectively, at the root of the tableau.

$\nvdash \mathcal{A}p \supset p$

$$
\begin{array}{c}
1.\ [\neg(\mathcal{A}p \supset p)]^1_2 \\
2.\ [\mathcal{A}p]^1_2 \\
3.\ [\neg p]^1_2 \\
4.\ [p]^1_1
\end{array}
$$

The general 2D-tableau remains open, and a countermodel can be read-off from the open branch where $W = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$, $\mathcal{M}^1_1 \vDash p$, and $\mathcal{M}^1_2 \nvDash p$. The following 2D-matrix illustrates the countermodel:

$$
\begin{pmatrix}
 & 1 & 2 \\
1 & p & \neg p \\
2 & \times & \times
\end{pmatrix}
$$

Since $\mathcal{M}^1_1 \vDash p$, it follows that $\mathcal{M}^1_1 \vDash \mathcal{A}p$, whence $\mathcal{M}^1_1 \vDash \mathcal{A}p \supset p$. On the other hand, since $\mathcal{M}^1_1 \vDash p$, it also follows that $\mathcal{M}^1_2 \vDash \mathcal{A}p$. But since $\mathcal{M}^1_2 \nvDash p$, we have $\mathcal{M}^1_2 \nvDash \mathcal{A}p \supset p$, whence $\mathcal{A}p \supset p$ is not generally valid.

4.1.2 Tableaux for diagonal validity

There is yet another notion of validity that is somewhat natural for two-dimensional modal logics, namely, diagonal validity. A sentence, φ, is *diagonally valid* if and only if φ holds at every pair $\langle w, w\rangle$ in every model.[36] As remarked in §2, when Davies and Humberstone offered a two-dimensional semantics, evaluating formulas with respect to pairs of worlds, they dropped a distinguished element from the models. This is because the intuitive function of representing the actual world was taken up by the upper world in the evaluation of formulas. So, rather than a single distinguished element defined in the models, we now have a large set of actual worlds, i.e. the distinguished element of each variant of the original model, or every upper index in every model. In fact, the novelty with respect to the fixedly operator is that it changes the model in question to a variant one, where a possibly different world becomes the actual one.

Although the two-dimensional notation of Davies and Humberstone simplifies this in a very elegant manner, it nevertheless leaves out an important aspect of the original account: once the fixedly operator takes effect, we are evaluating formulas relative to variants of the

[36] Humberstone (2008, p. 259–264), also discusses diagonal validity. Moreover, Kocurek (2016) adopts diagonal validity for a system containing actually and fixedly operators.

original model, where the distinguished element of the latter is supposed to represent intuitively the (only) real world, the other actual worlds in the variant models being alternatives with respect to it. In this sense, we can say that there is at most one actual world, even though different worlds might be considered as actual. Informally and intuitively, one of the insights that a distinguishedly operator brings back to two-dimensional modal logic is the following: that there is a real world in the space of two-dimensional possibilities to which we can always refer, even within contexts where we are far removed from it by embedding under modal or fixedly operators. *This is just what the actuality operator does in one-dimensional modal logic.* Another way to think of it is that distinguishedly marks the original model, bringing us back from any variant thereof.

In the absence of a distinguishedly operator, diagonal validity tells us that a sentence is valid just in case it holds at every world considered as actual. The sentence $\mathcal{A}\varphi \supset \varphi$, for example, happens to be both locally and diagonally valid. Also, there is a strong sense of necessity featuring in diagonal validity that was lacking in local validity. Since a sentence is diagonally valid just in case it holds at every pair along the diagonal, it holds at every possible world taken as actual. Admittedly, this is not the same as the intuitive notion of metaphysical necessity, which is represented in our 2D-matrices by truth along rows, and not truth along the diagonal. But it is a necessity nonetheless. In fact, diagonal validity brings to light precisely the notion of necessity defended by Davies and Humberstone, namely, $\mathcal{FA}$-necessity, whereas $\Box$-necessity usually corresponds to what is generally valid. Hence, even though we could not have $\mathcal{A}\varphi \supset \varphi$ to come out valid when validity was taken to correspond to $\Box$-necessary truth (general validity), it is valid when a different kind of necessity is in play.

Now, diagonal validity, or truth at every world taken as actual, can also be represented by diagonal tableaux as follows:

Definition 4.3 (Diagonal 2D-tableaux) A *diagonal 2D-tableau* is defined just like local 2D-tableaux, except that the root of the 2D-tableau is now the negation of the formula we want to prove, doubly-indexed by $\langle n, n\rangle$, where n is any index other than '0', i.e. $[\neg\varphi]^n_n$.

Diagonal 2D-tableaux, then, begin with any pair of identical indices, except that they are different from '0', corresponding thereby to an arbitrary point on the diagonal. We also define the notion of a 2D-tableau proof for the diagonal case as follows:

Definition 4.4 (Diagonal 2D-tableau proof) A *diagonal 2D-tableau proof* of a sentence, φ, is a closed diagonal 2D-tableau for $[\neg\varphi]^n_n$, where n is any index other than '0'.

Lastly, diagonal validity corresponds to a limiting case of diagonal consequence, where a sentence, φ, is a *diagonal consequence* of a set of sentences Γ if and only if for every model $\mathcal{M}$ and every pair $\langle w, w\rangle \in W$, if $\mathcal{M}^w_w \vDash \gamma$, for all $\gamma \in \Gamma$, then $\mathcal{M}^w_w \vDash \varphi$.

Now it can be verified by a tableau proof that $\mathcal{A}p \supset p$ is diagonally valid. We use the index '1' for n at the root of the tableau.

$\vdash \mathcal{A}p \supset p$

$$\begin{array}{c} 1.\ [\neg(\mathcal{A}p \supset p)]^1_1 \\ 2.\ [\mathcal{A}p]^1_1 \\ 3.\ [\neg p]^1_1 \\ 4.\ [p]^1_1 \\ \times \end{array}$$

The proof above suggests that local and diagonal validity coincide. However, the sentence $\odot\mathcal{A}p \supset p$ is locally, but not diagonally valid. In fact, this sentence also fails to be globally valid. Next we provide a failed tableau proof attempt of this sentence using diagonal 2D-tableaux:

$\nvDash \odot\mathcal{A}p \supset p$

$$\begin{array}{c} 1.\ [\neg(\odot\mathcal{A}p \supset p)]^1_1 \\ 2.\ [\odot\mathcal{A}p]^1_1 \\ 3.\ [\neg p]^1_1 \\ 4.\ [\mathcal{A}p]^0_1 \\ 5.\ [p]^0_0 \end{array}$$

The diagonal 2D-tableau remains open, and a countermodel can be read-off from the open branch where $W = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$, $\mathcal{M}^0_0 \vDash p$, and $\mathcal{M}^1_1 \nvDash p$. The following 2D-matrix illustrates the countermodel:

$$\begin{pmatrix} & 0 & 1 \\ 0 & p & \times \\ 1 & \times & \neg p \end{pmatrix}$$

Since $\mathcal{M}^0_0 \vDash p$, it follows that $\mathcal{M}^0_0 \vDash \mathcal{A}p$, and also $\mathcal{M}^0_0 \vDash \odot\mathcal{A}p$. Therefore, $\mathcal{M}^0_0 \vDash \odot\mathcal{A}p \supset p$. On the other hand, since $\mathcal{M}^0_0 \vDash p$, it also follows that $\mathcal{M}^0_1 \vDash \mathcal{A}p$, and $\mathcal{M}^1_1 \vDash \odot\mathcal{A}p$. But since $\mathcal{M}^1_1 \nvDash p$, we have $\mathcal{M}^1_1 \nvDash \odot\mathcal{A}p \supset p$, whence $\mathcal{A}p \supset p$ fails to hold at some point on the diagonal, and consequently it is not diagonally valid.

It is simple to check that local and diagonal validity coincide for distinguishedly-free sentences, for in the absence of $\odot$, there is no operator marking $w*$ in the language. Thus, if a sentence fails to be locally valid, it fails on some point on the diagonal in a model. On the other hand, if a sentence fails on some diagonal point in a model, if fails to be locally valid in the model generated by making that point the distinguished one.

5 Tableaux for Different Two-Dimensional Systems

The system $\mathbf{S5}_{2D}$ generalizes providing sound and complete tableaux for a variety of two-dimensional modal logics. The *locus classicus* of the logic $\mathbf{S5}\mathcal{A}$ is Crossley and Humberstone's seminal paper where a sound and complete axiomatization using general validity is offered for the propositional case. A Fitch-style natural deduction system for the same logic can be found in Hazen (1978), the only difference being that Hazen uses local rather than general validity. Now, by modulating between local and general tableaux one gets sound and complete systems for the logics defined by Hazen as well as Crossley and Humberstone, for $\mathbf{S5}_{2D}$ is clearly a conservative extension of $\mathbf{S5}\mathcal{A}$ with local validity, and our general tableaux are sound and complete too for $\mathbf{S5}\mathcal{A}$ with general validity.[37] Obviously, the same can be said for $\mathbf{S5}\mathcal{AF}$, the logic of the fixedly operator defined by Davies and Humberstone.

5.1 Tableaux for epistemic two-dimensional semantics

In this section we define a formal system based on the ideas of epistemic two-dimensional semantics as developed by Chalmers (1996, 2004). For this purpose, the system will be very similar to the two-dimensional logics for a priori knowledge considered in both Restall (2012) and Fritz (2013, 2014). The languages of these systems comprise truth-functional connectives, the modal operators $\Box$ and $\Diamond$, an actuality operator $\mathcal{A}$, and a primitive apriority operator, which we symbolize as $\mathcal{D}$, behaving similarly to $\mathcal{FA}$, i.e. it quantifies over every point on the diagonal. Fritz's logic is directly based on the epistemic two-dimensional semantics proposed by Chalmers, and so he suggests we should also define the dual of the a priori operator, which may be read intuitively as a conceivability operator as motivated by Chalmers[38] — even though Fritz himself does not commit to it being a formal rendering of conceivability.[39] In what follows we define the formal language and semantics of our system, which we call $\mathbf{S5E}_{2D}$. The main difference between $\mathbf{S5E}_{2D}$ and the logics of both Fritz and Restall[40] is that $\mathbf{S5E}_{2D}$ is extended to the first-order case, in which case we can introduce distinctions concerning rigidity.

Our language will be defined similarly to Definition 3.1, except that we make changes concerning the choice of two-dimensional operators:

Definition 5.1 (First-order Language) Let $\{c_1, c_2, ...\}$ be a set of *constant symbols*, $\{x_1, x_2, ...\}$ a set of *individual variables*, and $\{P_1^n, P_2^n, ...\}$ a set of n-place *predicate symbols* for each $n \in \mathbb{N}$. The terms t and formulas φ of $\mathcal{L}_{E2D}$ are recursively generated by the following grammar ($i, n \in \mathbb{N}$):

$$t ::= c_i \mid x_i$$

$$\varphi ::= P_i^n(t_1, ..., t_n) \mid t = t' \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \mathcal{A}\varphi \mid \mathcal{D}\varphi \mid \exists x_i \varphi$$

The basic formulas of the language comprise, again, atoms and their negations. Besides $\forall x_i \varphi$, $\Diamond$, and the other Boolean connectives, which are defined as usual, we also define $\mathcal{C}$, the dual of $\mathcal{D}$, as $\neg\mathcal{D}\neg$. Our models will be just as in Definition 3.2, except for having a $\mathcal{D}$-accessibility relation rather than an $\mathcal{F}$-accessibility relation:

Definition 5.2 (Constant domain 2D-centered models) A *constant domain 2D-centered model* is a tuple, $\mathcal{M} = \langle W, \langle w*, w* \rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{D}, \mathscr{D}, V \rangle$, such that

- $\mathcal{R}_\mathcal{D} \subseteq W \times W$, the $\mathcal{D}$-accessibility relation, is the least relation such that for every $v, w, z \in Z$, $\langle v, w \rangle \mathcal{R}_\mathcal{D} \langle z, z \rangle$.

[37] In appendix C we show some results concerning how $\mathbf{S5}_{2D}$ extends those systems.

[38] See Chalmers (2004), p. 219.

[39] See Fritz (2014), p. 386. Although Chalmers (2004, p. 219), motivates *conceivability* as the dual of *apriority*, see Chalmers (2011) for a criticism.

[40] These are equivalent. See Fritz (2014).

Again, this accessibility relation might be restricted in order to generate different systems. Here, we take it to be an equivalence relation. Moreover, this time we only impose the $\mathcal{R}_\Box$-rigidity condition on models. Even though a similar $\mathcal{R}_\mathcal{D}$-rigidity condition might be applied, the logic is intended to capture traditional features of epistemic two-dimensional semantics in which a sentence can be necessarily true without being a priori knowable. As we shall see below, having an $\mathcal{R}_\Box$-rigidity condition while lacking a similar $\mathcal{R}_\mathcal{D}$-rigidity condition gives us just what we want. Moreover, we also abandon (Neutrality), which gave us freedom with respect to which world was taken to be the actual one for basic formulas. We come back to this topic below.

We notice, nonetheless, that the logic in question does not contain explicit epistemic elements such as epistemic scenarios, for the two indices of evaluation are members of the same set of possible worlds. This, however, is just a simplification, and it is an assumption that both Restall and Fritz share in their systems. One may refer, then, to the upper index position intuitively as the *epistemic dimension*, or *scenarios*, and to the lower one as the *metaphysical dimension*, or possible worlds, in the spirit of Chalmers' two-dimensional semantics. Also, the distinguished element specified in the models may be dropped in the absence of a distinguishedly operator, in which case either diagonal or local validity might be naturally adopted — diagonal and local validity coincide for $\mathcal{L}_{E2D}$. The truth clauses in the semantics are very similar to what we already have in Definition 3.3, but we shall state those explicitly since $\mathbf{S5E}_{2D}$ involves, after all, a different language.

Definition 5.3 (Truth) For a pair $\langle v, w\rangle \in W$, and a valuation V in $\mathcal{M}$,

$$\begin{array}{lll}
\mathcal{M}^v_w \vDash P^n_i(t_1, ..., t_n) & \iff & \langle V(t_1, \langle v, w\rangle), ..., V(t_n, \langle v, w\rangle)\rangle \in V(P^n_i, \langle v, w\rangle);\\
\mathcal{M}^v_w \vDash t = t' & \iff & V(t, \langle v, w\rangle) = V(t', \langle v, w\rangle);\\
\mathcal{M}^v_w \vDash \neg\varphi & \iff & \mathcal{M}^v_w \nvDash \varphi\\
\mathcal{M}^v_w \vDash \varphi \wedge \psi & \iff & \mathcal{M}^v_w \vDash \varphi \text{ and } \mathcal{M}^v_w \vDash \psi;\\
\mathcal{M}^v_w \vDash \Box\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle\mathcal{R}_\Box\langle v, z\rangle, \text{ then } \mathcal{M}^v_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \Diamond\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle\mathcal{R}_\Box\langle v, z\rangle \text{ and } \mathcal{M}^v_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{A}\varphi & \iff & \mathcal{M}^v_v \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{D}\varphi & \iff & \text{for every } z \in Z, \text{ if } \langle v, w\rangle\mathcal{R}_\mathcal{D}\langle z, z\rangle, \text{ then } \mathcal{M}^z_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \mathcal{C}\varphi & \iff & \text{for some } z \in Z, \langle v, w\rangle\mathcal{R}_\mathcal{D}\langle z, z\rangle \text{ and } \mathcal{M}^z_z \vDash \varphi;\\
\mathcal{M}^v_w \vDash \forall x_i\varphi & \iff & \text{for every } x_i\text{-variant } V' \text{ of } V, \mathcal{M}^v_w \vDash \varphi[c_i/x_i];\\
\mathcal{M}^v_w \vDash \exists x_i\varphi & \iff & \text{for some } x_i\text{-variant } V' \text{ of } V, \mathcal{M}^v_w \vDash \varphi[c_i/x_i];
\end{array}$$

The notions of validity and consequence are defined as before.[41] Now, the tableau rules for $\mathbf{S5E}_{2D}$ are basically the ones we already have for the relevant portion of the language of $\mathbf{S5}_{2D}$. Notice that the second identity rule, the one involving substitution, can be assumed without any change in $\mathbf{S5E}_{2D}$. In fact, we could have presented a stronger rule for identity substitution in $\mathbf{S5}_{2D}$, involving change in both upper and lower indices, given the rigidity conditions imposed on the models, that is, if $[c = d]^i_n$ and $[\varphi(c)]^j_m$ already occur on the branch, then $[\varphi(d)]^j_m$ can be added to the same branch. This was not needed, however, given our (Neutrality) constraint and the upper exchange rule, in which case this variation of the identity substitution rule can be shown to be a derived rule:

[41] One can find the same semantics defined in Lampert (2017).

1. $[c = d]^i_n$
2. $[\varphi(c)]^j_m$
3. $[\varphi(c)]^i_m$
4. $[\varphi(d)]^i_m$
5. $[\varphi(d)]^j_m$

3 results from 2 by upper exchange, 4 follows from 1 and 3 by identity substitution, while 5 follows from 4 by upper exchange. Thus, our identity rules were fit to assume rigidity exclusively for $\mathcal{R}_\Box$-accessible worlds, whereupon they do not need to be modified for $\mathbf{S5E}_{2D}$. The only rule that needs, in effect, to be abandoned is the upper exchange rule, since we do not have (Neutrality) constraining the models for $\mathbf{S5E}_{2D}$. In the following we define the new rules for $\mathcal{D}$ and $\mathcal{C}$, where the former is classified as a necessity rule, and the latter as a possibility rule:

Definition 5.4 (Possibility Rules for $\mathcal{C}$)

If the index z is new to the branch,

$$\begin{array}{cc} [\mathcal{C}\varphi]^i_n & [\neg\mathcal{D}\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi]^z_z & [\neg\varphi]^z_z \end{array}$$

Definition 5.5 (Necessity Rule for $\mathcal{D}$)

For every index z occurring on the branch,

$$\begin{array}{cc} [\mathcal{D}\varphi]^i_n & [\neg\mathcal{C}\varphi]^i_n \\ \downarrow & \downarrow \\ [\varphi]^z_z & [\neg\varphi]^z_z \end{array}$$

The purpose of dropping (Neutrality) and the $\mathcal{R}_\mathcal{D}$-rigidity condition is that, in $\mathbf{S5E}_{2D}$, we are now able to find countermodels for $\Box(x = y) \supset \mathcal{D}(x = y)$. This is desirable in order to capture the view usually held by two-dimensionalists and exemplified by the Hesperus/Phosphorus case: although Hesperus is identical to Phosphorus, and necessarily so, this identity is not a priori knowable. We can take the first coordinate of a pair of worlds as determining the reference of a name just like a Fregean sense does. Because of this, it is desirable for names not to designate rigidly under the first dimension. The name 'Hesperus', for example, can be said to pick out Venus relative to a scenario where this is the brightest object in the evening sky. By contrast, if we consider a different scenario where the brightest object in the evening sky is Neptune, then 'Hesperus' picks out Neptune relative to this scenario.[42] Although 'Hesperus is Phosphorus' is necessarily true, it is not a priori since 'Hesperus' and 'Phosphorus' can pick out distinct entities relative to different scenarios — or first coordinates of members of W. This can be illustrated, formally, by the following tableau proof attempt:

$\nvdash \Box(a = b) \supset \mathcal{D}(a = b)$

[42] A more detailed exposition can be found in Chalmers (2004).

$$
\begin{array}{c}
1.\ [\neg(\Box(a=b)\supset\mathcal{D}(a=b))]^0_0\\
2.\ [\Box(a=b)]^0_0\\
3.\ [\neg\mathcal{D}(a=b)]^0_0\\
4.\ [\neg a=b]^1_1\\
5.\ [a=b]^0_0\\
6.\ [a=b]^0_1
\end{array}
$$

The tableau remains open, and a countermodel can be read-off from the open branch where $Z = \{0,1\}$, and thus $W = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$, $V(a, \langle 0,0\rangle) = V(b, \langle 0,0\rangle)$, and $V(a, \langle 0,1\rangle) = V(b, \langle 0,1\rangle)$. (Notice that this respects the $\mathcal{R}_\Box$-rigidity condition.) The following 2D-matrix illustrates the countermodel:

$$
\begin{pmatrix}
 & 0 & 1\\
0 & a=b & a=b\\
1 & \times & \times
\end{pmatrix}
$$

Since $\mathcal{M}^0_1 \vDash a = b$ and $\mathcal{M}^0_0 \vDash a = b$, $\mathcal{M}^0_0 \vDash \Box(a = b)$. However, since $\mathcal{M}^1_1 \nvDash a = b$, we have $\mathcal{M}^0_0 \nvDash \mathcal{D}(a = b)$, in which case $\nvDash \Box(a = b) \supset \mathcal{D}(a = b)$. Note, however, that if we assume (Neutrality) and upper index rigidity, we validate $\Box(a = b) \supset \mathcal{D}(a = b)$, for the upper exchange rule becomes valid as well, in which case the tableau above can be closed by an application of upper exchange to item 6, whereby we have 7. $[a = b]^1_1$, contradicting 4.

5.2 Semantic neutrality and the conceivability/possibility link

Now, suppose for a moment that we were to assume (Neutrality) in the semantics. This move validates the conceivability/possibility link, as illustrated below by a proof in the propositional portion of the language:

$\vdash \mathcal{C}p \equiv \Diamond p$

$$
\begin{array}{c}
1.\ [\neg(\mathcal{C}p\equiv\Diamond p)]^0_0
\end{array}
$$

$$
\begin{array}{cc}
2.\ [\neg(\mathcal{C}p\supset\Diamond p)]^0_0 & 3.\ [\neg(\Diamond p\supset\mathcal{C}p)]^0_0\\
4.\ [\mathcal{C}p]^0_0 & 10.\ [\Diamond p]^0_0\\
5.\ [\neg\Diamond p]^0_0 & 11.\ [\neg\mathcal{C}p]^0_0\\
6.\ [p]^1_1 & 12.\ [p]^0_1\\
7.\ [\neg p]^0_0 & 13.\ [\neg p]^0_0\\
8.\ [\neg p]^0_1 & 14.\ [\neg p]^1_1\\
9.\ [p]^0_1 & 15.\ [p]^1_1\\
\times & \times
\end{array}
$$

The use of upper exchange in the above (items 9 and 15) is licensed by (Neutrality). It also indicates something interesting about what is being proved in the logic, namely, that the truth of p is independent of the upper index. As we know, this feature is distinctive of

Davies and Humberstone's semantics, and it was adopted in the system $\mathbf{S5}_{2D}$ accordingly.[43] By and large, the link between $\mathcal{C}$ and $\Diamond$ is due to this, which fits nicely with the claim in Chalmers (2006, §3.5) that where φ is a *semantically neutral sentence*, φ is conceivable just in case it is also possible. According to Chalmers, "a semantically neutral expression is one whose extension in counterfactual worlds does not depend on how the actual world turns out." (p. 192) Some examples of semantically neutral expressions provided by Chalmers include 'and', 'consciousness', 'causal', 'philosopher', etc.. One can think of expressions of this kind as ones which are not Twin-Earthable, in contrast to most proper names, natural kind terms, and indexicals, for instance. A more formal account of semantic neutrality is also given by Fritz (2014, p. 410), where a sentence is semantically neutral in a model just in case "its truth is independent of the epistemic index", which translates to us as truth independent of the upper index. Thus, for semantically neutral sentences the upper exchange rule may be used freely, whereby a tight link between $\mathcal{C}$ and $\Diamond$ formulas is attained (analogously for $\mathcal{D}$ and $\Box$ formulas).[44]

A system for semantically neutral terms can be defined in the following lines. Let $\mathbf{S5EN}_{2D}$ be the system resulting from $\mathbf{S5E}_{2D}$ by constraining the models with (Neutrality) and adding the upper exchange rule to the stock of rules already in $\mathbf{S5E}_{2D}$. All the constants in the language will be semantically neutral. Additionally, we want these to involve a certain notion of *epistemic rigidity*, that is, rigidity with respect to the upper index, or first dimension. Although semantically neutral expressions are true or false independently of the upper index, we need to make sure the constants still denote the same entities whenever there is variation with respect to the upper index. For the system $\mathbf{S5}_{2D}$ this was accomplished under the guise of rigidity for $\mathcal{R}_{\mathcal{F}}$-accessible worlds. In $\mathbf{S5E}_{2D}$, however, we lack an accessibility relation that is exclusive for the first coordinates of the members of W. And even though we could still define rigidity for semantically neutral constants with respect to every first coordinate of every pair by stipulation, we can also make use of a more general notion of *super-rigidity*, which is also employed by Chalmers (2012, p. 370), besides being closely related to semantic neutrality.[45] The notion of super-rigidity can be defined formally by stipulating that for any constant symbol c_i in the language of $\mathbf{S5EN}_{2D}$,

$$\text{(Super-rigidity) For every } u, v, w, z \in Z,\ V(c_i, \langle u, v\rangle) = V(c_i, \langle w, z\rangle).$$

Super-rigidity, therefore, makes semantically neutral constants rigid with respect to both upper and lower indices of evaluation.[46] It is also a simple generalization of both $\mathcal{R}_{\Box}$ and $\mathcal{R}_{\mathcal{F}}$ rigidities. Such constants will denote the same entities at any point in a two-dimensional model, whereupon they will not be susceptible of Twin-Earthability. $\mathbf{S5EN}_{2D}$ is, consequently, a logic for semantically neutral terms. It is also a subsystem of $\mathbf{S5}_{2D}$, since $\mathcal{D}$ can be defined as $\mathcal{FA}$, whence $\mathbf{S5EN}_{2D}$ is sound and complete as well. Thus, in effect, we

[43] Although, of course, it might be dropped, generating thereby a different set of validities.

[44] It is a simple matter to check that for any $\mathcal{A}$-free formula φ, we have both $\Diamond\varphi \equiv \mathcal{C}\varphi$ and $\Box\varphi \equiv \mathcal{D}\varphi$, the latter corresponding to axiom $\mathcal{F}6$ in Davies and Humberstone's $\mathbf{S5}\mathcal{AF}$. See appendix C.

[45] Chalmers thinks all super-rigid expressions are semantically rigid. Although there are some wrinkles in the converse claim, if a semantically neutral expression is not equivalent to a super-rigid expression, it is at least equivalent to a compound of super-rigid expressions. For more details, see Chalmers (2012, p. 370).

[46] In Chalmers' terms, this kind of expression has a constant two-dimensional intension. See Chalmers (2012, p. 370).

already had in $\mathbf{S5}_{2D}$ a comprehensive system for semantically neutrality, except for lacking primitive apriority operators. The system herein presented, therefore, delivers what one would expect when it comes to a logic for conceivability/possibility based on a semantics such as the one originally proposed by Chalmers, as long as one imposes (Neutrality) on the models, substantiated by appropriate rigidity conditions and the upper exchange rule. Finally, the same modularity can be achieved regarding validity in the tableaux for both $\mathbf{S5E}_{2D}$ and $\mathbf{S5EN}_{2D}$.

6 Coda: Inexpressibility, necessity, and multidimensionality

Before closing, let us come back to the topic of expressive incompleteness in modal languages, which was briefly addressed by examples in the beginning of this paper, as well as to the relations between actuality, necessity, and the a priori. Many of the issues in this section should be taken as exploratory, with several open questions for future research, both formal and philosophical. Nonetheless, we must acknowledge some of the pressing philosophical questions that arise concerning different actuality operators, to wit, rigid and non-rigid, in the context of two-dimensional modal logics, and how they relate to the two notions of necessity ($\Box$ and $\mathcal{FA}/\mathcal{D}$) at work in the systems defined above. In particular, we focus on the system $\mathbf{S5}_{2D}$, although similar issues apply, *mutatis mutandis*, to $\mathbf{S5E}_{2D}$.

The introduction of a significant number of operators in the language of $\mathbf{S5}_{2D}$ was motivated by the claim that basic modal languages, whose modal operators include only $\Box$ and $\Diamond$, are expressively incomplete, that is, they are not sufficiently powerful in order to fully represent modal discourse. In particular, this was the reason for introducing $\mathcal{A}$, $\odot$, and $\otimes$. We have started with Crossley and Humberstone's sentence (1), the inexpressibility of which has motivated the introduction of $\mathcal{A}$ into a first-order modal language, and then we offered a two-dimensional generalization of (1), namely, (3), which was in turn claimed to be inexpressible in a first-order two-dimensional language, thereby motivating the addition of $\odot$ into our stock of operators. Similar considerations have underpinned the adoption of $\otimes$ as well.

At this point, however, one cannot but question whether there would not be similar inexpressibility issues in even more powerful languages, such as the very same claimed to be sufficiently rich in order to formalize (3), for example. After all, do we have reason to believe our current suite of modal operators to be appropriate in order to represent the kind of modal discourse we are interested here?

As we shall now see, any adequate answer to this bears upon what is meant by the appropriateness of a formal language, or a logical system itself. While there is certainly a question regarding the expressive completeness of a language such as $\mathcal{L}_{2D}$, this is the kind of result deserving a different paper. By contrast, one might also ask about whether the formal language or system at hand is appropriate in the sense of being ultimately philosophically adequate. This notion of adequacy, in turn, can be clarified with a couple of examples. The modal logic $\mathbf{T}$ is characterized by the axiom $\Box\varphi \supset \varphi$, whence it seems to be inadequate as a logic for obligation: it is simply not the case that whenever one ought to do p, then p. On the other hand, it is usually assumed that knowledge is factive, that is, if one knows that p, then p, whence **T** is sometimes claimed to be adequate as a logic for knowledge.[47] Another, familiar cases, whose details we skip, are Salmon's (1989) argument based on the essentiality of origin to the effect that modal logics containing the **S4** axiom $\Box\varphi \supset \Box\Box\varphi$ are inadequate in order to represent our intuitive notion of (metaphysical) necessity, as well as Dummett's argument that metaphysical modality does not obey the **B** axiom $\varphi \supset \Box\Diamond\varphi$. The former denies that metaphysical necessity is transitive, while the latter denies it is symmetric. Hence, both reject **S5** as an adequate logic of necessity.[48] Likewise, we saw in §2.1 that $\mathcal{F}$ was introduced into modal languages with an actuality operator because of the intuitive invalidity of $\mathcal{A}\varphi \supset \Box\mathcal{A}$, for, as Humberstone (2004, p. 21) points out, "to many there seems to be a sense in which what is actually the case need not be necessarily actually the case," whence the alternative notion of necessity delivered by $\mathcal{FA}$. While a basic modal language for **S5** is expressively deficient, one containing $\mathcal{A}$ might be claimed to be, in a sense, philosophically impaired. The question we face, therefore, is about the philosophical adequacy of the systems herein defined.

Regarding this matter we now observe that the introduction of a distinguishedly operator in a language for two-dimensional modal logic comes not without a cost. Despite being needed on pain of expressive weakness, the addition of $\odot$ results in the formula $\odot\mathcal{A}\varphi \supset \mathcal{FA} \odot \mathcal{A}\varphi$ being valid in $\mathbf{S5}_{2D}$.[49] This formula is, in effect, a two-dimensional analogue of the already much discussed formula $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$. Consequently, if something is distinguishedly actually true, it is deeply necessary that it is so, in which case the distinguishedly operator seems to subvert the very motivation for moving to a two-dimensional modal logic! Furthermore, notice that the role of $\odot\mathcal{A}$ in this case is analogous to the one of a rigid $\mathcal{A}$ in one-dimensional modal logic. Notwithstanding its non-rigidity in a two-dimensional framework, $\mathcal{A}$ plays the role of a distinguished actuality operator in $\mathbf{S5}\mathcal{A}$. Therefore, the intuition that it is a contingent matter which world is the actual one should be understood in terms of rigid actuality: despite being superficially necessary, it is deeply contingent which world is the distinguished actual world, whence the thought that $\odot\mathcal{A}\varphi \supset \mathcal{FA}\odot\mathcal{A}\varphi$ is, too, intuitively invalid. In the midst of repairing the expressive deficiency of two-dimensional languages, it looks as though we throw the baby out with the bathwater.

Not surprisingly, this problem may be solved by the addition of a third index of evaluation and a new necessity operator, herein symbolized by $\mathcal{N}$:

$$\mathcal{M}^{\,u}_{\substack{v\\w}} \vDash \mathcal{N}\varphi \text{ if and only if for every } z \in Z,\ \mathcal{M}^{\,z}_{\substack{v\\w}} \vDash \varphi.$$

We can also define the dual of $\mathcal{N}$, symbolized by $\mathcal{P}$, as $\neg\mathcal{N}\neg$. Models for propositional three-dimensional modal logic can be defined by letting W be $Z \times Z \times Z$, for some set Z, where $\langle w*, w*, w* \rangle$ is a distinguished element of W, $\mathcal{R}_\Box$, $\mathcal{R}_\mathcal{F}$, and $\mathcal{R}_\mathcal{N}$ are accessibility relations corresponding to $\Box$, $\mathcal{F}$, and $\mathcal{N}$ formulas, and V is a function from propositional letters to members of W, which are now ordered triples $\langle u, v, w \rangle$. To be more explicit, the semantic

[47] Williamson (2013), for example, argues that **T** is the correct system for a logic of knowledge.

[48] Ultimately, these questions amount to the kind of accessibility relation one wants for a basic modal system.

[49] This holds regardless of the chosen notion of validity — local, general, or diagonal.

clauses can be defined as follows:

$$
\begin{array}{lll}
\mathcal{M}_{\substack{u\\v\\w}} \vDash p & \Longleftrightarrow & \langle u,v,w\rangle \in V(p);\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \neg\varphi & \Longleftrightarrow & \mathcal{M}_{\substack{u\\v\\w}} \nvDash \varphi\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \varphi \wedge \psi & \Longleftrightarrow & \mathcal{M}_{\substack{u\\v\\w}} \vDash \varphi \text{ and } \mathcal{M}_{\substack{u\\v\\w}} \vDash \psi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \Box\varphi & \Longleftrightarrow & \text{for every } z \in Z, \text{ if } \langle u,v,w\rangle \mathcal{R}_{\Box} \langle u,v,z\rangle, \text{ then } \mathcal{M}_{\substack{u\\v\\z}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \Diamond\varphi & \Longleftrightarrow & \text{for some } z \in Z, \langle u,v,w\rangle \mathcal{R}_{\Box} \langle u,v,z\rangle \text{ and } \mathcal{M}_{\substack{u\\v\\z}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \mathcal{A}\varphi & \Longleftrightarrow & \mathcal{M}_{\substack{u\\v\\v}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \odot\varphi & \Longleftrightarrow & \mathcal{M}_{\substack{u\\u\\u}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \otimes\varphi & \Longleftrightarrow & \mathcal{M}_{\substack{u\\w\\w}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \mathcal{F}\varphi & \Longleftrightarrow & \text{for every } z \in Z, \text{ if } \langle u,v,w\rangle \mathcal{R}_{\mathcal{F}} \langle u,z,w\rangle, \text{ then } \mathcal{M}_{\substack{u\\z\\w}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \mathcal{S}\varphi & \Longleftrightarrow & \text{for some } z \in Z, \langle u,v,w\rangle \mathcal{R}_{\mathcal{F}} \langle u,z,w\rangle \text{ and } \mathcal{M}_{\substack{u\\z\\w}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \mathcal{N}\varphi & \Longleftrightarrow & \text{for every } z \in Z, \text{ if } \langle u,v,w\rangle \mathcal{R}_{\mathcal{N}} \langle z,v,w\rangle, \text{ then } \mathcal{M}_{\substack{z\\v\\w}} \vDash \varphi;\\
\mathcal{M}_{\substack{u\\v\\w}} \vDash \mathcal{P}\varphi & \Longleftrightarrow & \text{for some } z \in Z, \langle u,v,w\rangle \mathcal{R}_{\mathcal{N}} \langle z,v,w\rangle \text{ and } \mathcal{M}_{\substack{z\\v\\w}} \vDash \varphi.
\end{array}
$$

The notions of truth, consequence, and validity, can be appropriately defined by generalizing the corresponding notions in the two-dimensional case. For example, we can define local validity as truth at $\langle w*, w*, w*\rangle$ in every model, general validity as truth at every pair $\langle u, v, w\rangle$ in every model, or even, say, *three-dimensional validity*, as truth at every triple $\langle w, w, w\rangle$ in every model. Notice that given the semantics just outlined, although we still have the validity of $\odot\mathcal{A}\varphi \supset \mathcal{FA} \odot \mathcal{A}\varphi$, the formula $\odot\mathcal{A}\varphi \supset \mathcal{N} \odot \odot\mathcal{A}\varphi$ is not valid, as desired.[50]

Now there are several questions about the philosophical prospects of a three-dimensional modal logic based on the semantics above, which directly affect its status as a plausible solution to the problem under consideration.[51] Does $\mathcal{N}$ in itself define any interesting notion of necessity?[52] And what about the compound $\mathcal{N}\odot$ delivering truth at every triple $\langle z, z, z\rangle$? Does that in turn correspond to any intuitive or useful notion of necessity? Since distinguishedly actual truths, despite being deeply necessary, are not necessary in the sense of $\mathcal{N}\odot$, this invites the question about whether it is the latter, then, instead of $\mathcal{FA}$, that best

[50]In both cases, "validity" means either local, general, or three-dimensional.

[51]Tableaux for three-dimensional modal logic can be easily defined by extending the two-dimensional case with another numeric index. Note that we can also define a three-dimensional version of $\otimes$, whereby the third index of evaluation is copied up to the first one. If we symbolize this as $\oplus$, its semantics can be defined as $\mathcal{M}_{\substack{u\\v\\w}} \vDash \oplus\varphi$ if and only if $\mathcal{M}_{\substack{w\\w\\w}} \vDash \varphi$. More generally, for any dimension n, we can define new actuality and *Ref* operators accordingly. The same is true, of course, for the other two-dimensional operators mentioned here.

[52]The answer should be negative in case a condition similar to (Neutrality) is assumed:

(3D-Neutrality) If φ is a basic formula, then for every $t, u, v, w, z \in Z$, $\mathcal{M}_{\substack{u\\v\\w}} \vDash \varphi$ if and only if $\mathcal{M}_{\substack{t\\z\\w}} \vDash \varphi$.

Once (3D-Neutrality) is assumed, the models validate $\varphi \equiv \mathcal{N}\varphi$ for every basic formula φ, whence $\mathcal{N}$ does not represent any notion of necessity in itself.

represents the intuitive notion of necessity like the one that have motivated the introduction of $\mathcal{F}$.

In spite of all this, the attentive reader will notice that $\odot$ is not rigid in this three-dimensional language anymore: it is a copy down operator, just like $\mathcal{A}$. This 'de-rigidification' serves to illustrate that in a three-dimensional language lacking a rigid actuality operator there will be no formal representation of a sentence equivalent to the following truth conditions: where Σ is an existential quantifier over members of W,

(9) $\Sigma\langle w, w, w\rangle \forall x(Rx\text{-}at\text{-}\langle w*, w*, w*\rangle \supset Sx\text{-}at\text{-}\langle w, w, w\rangle)$.

The best we could do here would be to formalize (9) as (10):

(10) $\mathcal{P} \odot \forall x(\odot Rx \supset Sx)$.

However, $\odot$ will not take us to the distinguished triple $\langle w*, w*, w*\rangle$ as desired. This problem is already familiar, whence we can solve it by likewise familiar methods: a new actuality operator, $\mathcal{B}$, can be added to the language, pointing invariably to $w*$ in Z:

$$\mathcal{M}^{\,u}_{\substack{v\\w}} \vDash \mathcal{B}\varphi \text{ if and only if } \mathcal{M}^{\,w*}_{\substack{v\\w}} \vDash \varphi,$$

in which case (9) can now be formalized as

(11) $\mathcal{P} \odot \forall x(\mathcal{B} \odot Rx \supset Sx)$.

Is there an English reading available for (11)? Unless there are natural readings for three-dimensional operators, we do not have any expectations of finding one. Besides, there is an element of artificiality in three-dimensional operators like the ones just defined: apart from the fact that they do not seem to have any natural-language readings, there does not seem to be philosophical or intuitive interpretations for them as well unless one takes $\mathcal{N}\odot$, for instance, as a deep necessity operator, which leaves $\mathcal{FA}$ unexplained. Maybe some natural interpretation will appear that makes multidimensional necessities more intuitive. At the moment, however, they strike us as rather puzzling. This might cause one to raise an eyebrow and suspect that, even if we could make intuitive sense of such a thing like a three-dimensional modal logic, it seems to be much more the product of logical curiosity than anything else.

Nevertheless, a three-dimensional system helps to illustrate just how the problems motivating the introduction of a new dimension to modal logics, as well as new actuality operators, are more general than we once thought them to be, besides being interestingly related: on pain of expressive deficiency, we need rigid actuality operators in modal languages; yet, because they are rigid, they will inhibit whatever necessity operators we have, shielding, as it were, the formulas within their scope, thereby validating that rigidly actual truths are necessarily so. If, on the one hand, we accept the latter, the philosophical motivation to introduce a fixedly operator seems to fall flat. On the other hand, by adding more dimensions and operators there is no longer a principled reason, so it seems, to endorse that deep necessity is represented by $\mathcal{FA}$; as we have just shown, we might as well take $\mathcal{N}\odot$ to be our deep necessity operator, since rigidly actual truths turn out to be $\mathcal{FA}$-necessary despite the intuition that they are (deeply) contingent. Once, however, we have the new (rigid) actuality operator $\mathcal{B}$ in our three-dimensional language, it is an easy matter to see that $\mathcal{B}\odot\varphi \supset \mathcal{N}\odot\mathcal{B}\odot\varphi$ also becomes valid, which we might take in turn as reason to define a four-dimensional modal logic, and so on. Now, of course, we are back to square one, and the problem appears to recur in any dimension whatsoever.

On the two-dimensionalist's behalf, it might be claimed that some of these issues are purely formal, especially the ones involving three-dimensional logics or more. However, this does not change the fact that an explanation for the validity of $\odot\mathcal{A}\varphi \supset \mathcal{FA}\odot\mathcal{A}\varphi$ is needed if we are supposed to identify $\mathcal{FA}$ as a deep necessity operator. After all, *there seems to be a sense in which what is distinguishedly actually the case need not be deeply necessarily distinguishedly actually the case*. In any event, it seems clear enough that $\mathcal{FA}$ defines a notion of necessity, but far from clear it is a *deep* one; whatever it is, it just might not be what Evans, let alone Davies and Humberstone, had in mind.[53]

7 Conclusion

We have shown tableau calculi for first-order two-dimensional modal logics to be sound and complete. The methods in use here present themselves as simple generalizations of familiar tools for first-order modal logics, and there seems to be no special difficulties in generalizing them for more dimensions as well. Furthermore, we have motivated a new operator, called "distinguishedly", that brings to light a formal distinction between rigid and non-rigid actuality operators in two-dimensional languages. Not only that, but we have also observed that it might give rise to unpleasant philosophical consequences for two-dimensional modal logic itself. In a nutshell: despite being well-motivated, the philosophical interpretation of two-dimensional operators is far from settled, especially the necessity attributed to $\mathcal{FA}$ that has long been part of philosophical lore. We wish we had more definitive things to say regarding these matters, though it is worth noting that such issues do not affect the tableaux developed here *qua* proof systems, but only the intuitive interpretation of two-dimensional operators. For the moment we cannot avoid but closing this paper with several open questions to which we now add the following:

- Call a Kripke model 'deterministic' if it validates formulas of the kind $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$. Can Kripke models for languages that are not expressively deficient in the sense above (i.e. containing actuality operators) be defined in non-deterministic ways?

- What should we make of the claim that deep necessity and/or a priori knowledge corresponds to truth along the diagonal once our models validate formulas such as $\odot\mathcal{A}\varphi \supset \mathcal{FA}\odot\mathcal{A}\varphi$? Are the issues raised here a problem for different brands of two-dimensionalism such as the versions developed by Kaplan, Stalnaker, Jackson, and Chalmers?

- There are interesting questions regarding the epistemic profile of rigid and non-rigid actual truths. While $\mathcal{A}\varphi \equiv \varphi$ is $\mathcal{FA}$-necessary and usually taken to be a priori knowable (Evans (1979, p. 200), Davies (2004, p. 100)), $\odot\mathcal{A}\varphi \equiv \varphi$ is not $\mathcal{FA}$-necessary. In this sense, the thought that grass is actually green seems radically different from

[53] In Lampert (2017) we point out similar issues involving the apriority operator in a semantics based on epistemic two-dimensional semantics, as developed by Chalmers.

the corresponding thought that grass is distinguishedly actually green, although the formulas just mentioned only differ when occurring embedded in $\mathcal{F}$ contexts.

- The formula $\odot\mathcal{A}\varphi \supset \varphi$ is locally, but it is neither diagonally nor generally valid. If a local account of validity is preferred, though, this means that some valid formulas do not hold along the diagonal. This brings complications to the traditional view that diagonal necessity corresponds to apriority. Is this a reason for rejecting local validity for two-dimensional modal logics, or diagonal necessity as corresponding to apriority?

- The formula $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ holds in normal systems of modal logic from **K** up to **S5** endowed with an actuality operator. Thus, it also holds in systems where the accessibility relation is restricted at most by seriality, such as the system **D**, standardly assumed for deontic modality. Moreover, there are also reasons for adding actuality operators to modal languages for deontic logic (Humberstone (1982)). If so, is $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$ appropriate once we interpret $\Box$ in a deontic fashion?

- Humberstone (1982) distinguishes between actuality and subjunctive operators on the grounds that the former inhibits while the latter activates modal operators. This distinction becomes obscured once we consider multidimensional languages. $\mathcal{A}$ inhibits only the basic modal operators $\Box$ and $\Diamond$, but it is binded by $\mathcal{F}$ and $\mathcal{D}$. By contrast, $\odot\mathcal{A}$ inhibits all of these. Are there similar issues with subjunctive operators? Moreover, how does the issues discussed here concerning different actuality operators affect modal systems with subjunctive markers, such as the ones defined by Wehmeier (2004, 2005), in which an actuality operator is used as a tool for mood distinctions?

- Concerning proof systems, indexed tableaux seem to generalize nicely for multidimensional modal logics including several kinds of operators. There appears to be no impediment, at least in principle, to generalize indexed tableaux for n-dimensional modal logics. What about different proof systems such as hypersequents or natural deduction? For instance, could Fitch-style natural deduction systems be defined for multidimensional modal logics?

A Appendix

A.1 Soundness and Completeness for $\mathbf{S5}_{2D}$

The soundness theorem for $\mathbf{S5}_{2D}$ is an extension of corresponding proofs for modal logic presented, for instance, in Fitting and Mendelsohn (1998) and Priest (2008).

Definition A.1 (Satisfiability) Let S be a set of doubly-indexed formulas. We say S is *satisfiable* in $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$ just in case there is a function, f, assigning to each (single) index n occurring in S a possible world $f(n) \in Z$, where $f(0) = w*$, $W = Z \times Z$, such that,

- If $[\varphi]^i_n \in S$, then φ is true at $f(n)$ relative to $f(i)$, i.e. $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$.
- If the pairs of indices $\langle i, n\rangle$ and $\langle i, m\rangle$ are in S, then $\langle f(i), f(n)\rangle \mathcal{R}_{\Box} \langle f(i), f(m)\rangle$.
- If the pairs of indices $\langle i, n\rangle$ and $\langle j, n\rangle$ are in S, then $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle f(j), f(n)\rangle$.

Definition A.2 A tableau branch $\mathfrak{b}$ is *satisfiable* just in case the set of doubly-indexed formulas on it is satisfiable in some model, and a tableau is *satisfiable* just in case some branch of it is satisfiable.

Lemma A.1 *A closed tableau is not satisfiable.*

Proof. Suppose that $\mathcal{T}$ is a closed tableau that is also satisfiable. By Definition A.2, some branch $\mathfrak{b}$ of $\mathcal{T}$ is satisfiable, whence there is a model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, satisfying the set S of doubly-indexed formulas on $\mathfrak{b}$ by way of the function f, by Definition A.1. But since, by assumption, $\mathcal{T}$ is also closed, for some formula φ and pair of indices $\langle i, n\rangle$, $[\varphi]^i_n \in S$ and $[\neg\varphi]^i_n \in S$. Hence, by Definition A.1, $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$ and $\mathcal{M}^{f(i)}_{f(n)} \nvDash \varphi$, which is impossible. □

Lemma A.2 *If one of the rules in $\mathbf{S5}_{2D}$ is applied to a satisfiable 2D-tableau, it results in another satisfiable 2D-tableau.*

Proof. Suppose $\mathfrak{b}$ is a branch of a satisfiable 2D-tableau $\mathcal{T}$ and that we apply one of the rules in $\mathbf{S5}_{2D}$ to a doubly-indexed formula occurring on $\mathfrak{b}$. It is easily seen that the result is another satisfiable tableau. Since, by assumption, $\mathcal{T}$ is satisfiable, by Definition A.2 at least one of its branches, say, $\mathfrak{b}*$, is satisfiable. Now, if $\mathfrak{b}* \neq \mathfrak{b}$, then $\mathfrak{b}*$ remains satisfiable after we apply a rule on $\mathfrak{b}$, whence we have a satisfiable 2D-tableau by Definition A.2. On the other hand, if $\mathfrak{b}* = \mathfrak{b}$, we need to consider several cases. We omit the arguments for the Booleans, modal operators, quantifiers, and identity, since these are routine.[54] In order to carry on such arguments in the 2D framework one just needs to add a pair of indices where the first coordinate does not do any work.

Suppose $[\mathcal{A}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{A}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{A}\varphi$, by Definition A.1. Thus, $\mathcal{M}^{f(i)}_{f(i)} \vDash \varphi$, by the truth clause for $\mathcal{A}$. Therefore, applying

[54]See, for instance, Fitting and Mendelsohn (1998, pp. 58–9, 122–123), and Priest (2008, pp. 31–2, 322–323).

the rules for $\mathcal{A}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\mathcal{A}$ is analogous.

Suppose $[\odot\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\odot$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \odot\varphi$, by Definition A.1. Thus, $\mathcal{M}^{w*}_{f(n)} \vDash \varphi$, by the truth clause for $\odot$, in which case $\mathcal{M}^{f(0)}_{f(n)} \vDash \varphi$, since '0' is fixed to $w*$. Therefore, applying the rules for $\odot$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\odot$ is analogous.

Suppose $[\otimes\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\otimes$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \otimes\varphi$, by Definition A.1. Thus, $\mathcal{M}^{f(n)}_{f(n)} \vDash \varphi$, by the truth clause for $\otimes$. Therefore, applying the rules for $\otimes$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. The argument for $\neg\otimes$ is analogous.

Suppose $[\mathcal{F}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{F}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{F}\varphi$, by Definition A.1. Also, for every pair of indices $\langle i, n\rangle$ and $\langle j, n\rangle$ occurring on $\mathfrak{b}$, $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle f(j), f(n)\rangle$. Thus, $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$, by the truth clause for $\mathcal{F}$. On the other hand, suppose $[\neg\mathcal{S}\varphi]^i_n$ occurs on $\mathfrak{b}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \neg\mathcal{S}\varphi$, by Definition A.1, and so $\mathcal{M}^{f(i)}_{f(n)} \nvDash \mathcal{S}\varphi$. Hence, for every $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle f(j), f(n)\rangle$, $\mathcal{M}^{f(j)}_{f(n)} \nvDash \varphi$, by the truth clause for $\mathcal{S}$, and so $\mathcal{M}^{f(j)}_{f(n)} \vDash \neg\varphi$. Therefore, applying the rules for $\mathcal{F}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau.

Suppose $[\mathcal{S}\varphi]^i_n$ occurs on $\mathfrak{b}$, and we apply the rule for $\mathcal{S}$. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \mathcal{S}\varphi$, by Definition A.1. Thus, for some $z \in Z$, $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{F}} \langle z, f(n)\rangle$ and $\mathcal{M}^{z}_{f(n)} \vDash \varphi$, by the truth clause for $\mathcal{S}$. Let g be like f except that $g(j) = z$, where j is a new index occurring on $\mathfrak{b}$. Since g and f agree except for j, the tableau extended by g is satisfiable. Moreover, by definition of g, $\langle g(i), g(n)\rangle \mathcal{R}_{\mathcal{F}} \langle g(j), g(n)\rangle$ and $\mathcal{M}^{g(j)}_{g(n)} \vDash \varphi$. By choice of g, it follows that $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$. The argument for $\neg\mathcal{F}$ is analogous. Therefore, applying the rules for $\mathcal{S}$ to a satisfiable 2D-tableau results in another satisfiable 2D-tableau.

Finally, let φ be a basic formula and suppose $[\varphi]^i_n$ occurs on $\mathfrak{b}$, and that we apply the upper exchange rule. Since $\mathfrak{b}$ is satisfiable, $\mathcal{M}^{f(i)}_{f(n)} \vDash \varphi$, by Definition A.1. Moreover, since φ is a basic formula, $\mathcal{M}^{f(j)}_{f(n)} \vDash \varphi$, for any first coordinate $f(j)$ of any pair of worlds, given that the models are constrained by (Neutrality). Therefore, applying the upper exchange rule to a satisfiable 2D-tableau results in another satisfiable 2D-tableau. □

Theorem A.1 (Soundness) *If φ has a 2D-tableau proof, then φ is valid.*

Proof. Suppose φ has a 2D-tableau proof, in which case there is a closed 2D-tableau, $\mathcal{T}$, beginning with $[\neg\varphi]^0_0$. For a contradiction, assume that φ is not valid. Thus, there is a 2D-centered model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, such that $\mathcal{M}^{w*}_{w*} \nvDash \varphi$. Let f be a function such that $f(0) = w*$. By Definition A.1, $\{[\neg\varphi]^0_0\}$ is satisfiable. Moreover, since its only branch is satisfiable, $\mathcal{T}$ is also satisfiable, and so is any 2D-tableau extending it by way of the rules in $\mathbf{S5}_{2D}$, by Lemma A.2. Therefore, $\mathcal{T}$ is both closed and satisfiable, contradicting Lemma A.1, whence $\mathcal{M}^{w*}_{w*} \vDash \varphi$. □

Infinite Tableaux and Systematic Procedure. Some 2D-tableaux may run infinitely, and by König's Lemma these will have at least one infinite branch. In what follows we give an example of a failed proof attempt in $\mathbf{S5}_{2D}$ generating an infinite 2D-tableau:

1. $[\neg\mathcal{S}(p\supset\Box p)]^0_0$
2. $[\neg(p\supset\Box p)]^0_0$
3. $[p]^0_0$
4. $[\neg\Box p]^0_0$
5. $[\neg p]^0_1$
6. $[\neg(p\supset\Box p)]^1_0$
7. $[p]^1_0$
8. $[\neg\Box p]^1_0$
9. $[\neg p]^1_2$

$\vdots$

A 2D-matrix for the above looks like this:

$$\begin{pmatrix} & 0 & 1 & 2 & 3 & \dots \\ 0 & p & \neg p & \times & \times & \\ 1 & p & \times & \neg p & \times & \\ 2 & p & \times & \times & \neg p & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

Using the same procedure as before to devise a countermodel we have the following:

> Since $\mathcal{M}^0_0 \vDash p$ and $\mathcal{M}^0_1 \nvDash p$, it follows that $\mathcal{M}^0_0 \nvDash \Box p$. Hence, $\mathcal{M}^0_0 \nvDash p \supset \Box p$.
>
> Since $\mathcal{M}^1_0 \vDash p$ and $\mathcal{M}^1_2 \nvDash p$, it follows that $\mathcal{M}^1_1 \nvDash \Box p$. Hence, $\mathcal{M}^1_0 \nvDash p \supset \Box p$.
>
> Since $\mathcal{M}^2_0 \vDash p$ and $\mathcal{M}^2_3 \nvDash p$, it follows that $\mathcal{M}^2_2 \nvDash \Box p$. Hence, $\mathcal{M}^2_0 \nvDash p \supset \Box p$.
>
> ... Hence, $\mathcal{M}^0_0 \nvDash \mathcal{S}(p \supset \Box p)$.

Even though a finite countermodel can be easily constructed to invalidate $\mathcal{S}(p \supset \Box p)$,[55] we need a *systematic procedure* for constructing 2D-tableaux in order to prove that any tableau so constructed is such that, if it has an open branch, finite or not, there is a countermodel for it.

Now, several systematic procedures of this kind can be found in the literature, even for the modal cases. In particular, we direct the reader again to Fitting and Mendelsohn (1998, pp. 126–7,)[56] since our strategy will be based on it. Thus, let $[\varphi]^i_n$ be any doubly-indexed formula of which we want to know whether or not it is satisfiable. For stage $n = 1$, introduce $[\neg\varphi]^i_n$ to the 2D-tableau as line 1. Next, suppose n stages have been completed. If the 2D-tableau is already closed, then we have produced a proof of $[\varphi]^i_n$. On the other hand, if the 2D-tableau remains open, then we proceed to stage $n + 1$ in which we take an open branch $\mathfrak{b}$ of the 2D-tableau such that $\mathfrak{b}$ is the leftmost highest point on it, and for each doubly-indexed formula $[\psi]^j_m$[57] such that it occurs on $\mathfrak{b}$ we extend the 2D-tableau as follows:

[55] We leave this case to the reader.

[56] Smullyan (1995, pp. 58–9), presents one for first-order logic.

[57] Where the indices i and n may be identical to j and m, respectively.

I If $[\psi]^j_m$ is a basic formula, say, $[\chi]^k_p$, add $[\chi]^l_p$ to the end of $\mathfrak{b}$, where l is an upper index added by either $\mathcal{S}$ or $\mathcal{F}$ (if it does not already contain it).

II If a constant, say, c, is on $\mathfrak{b}$, add $[(c = c)]^k_p$ to the end of $\mathfrak{b}$ for all pairs of indices $\langle k, p\rangle$ occurring on $\mathfrak{b}$.

III If the pair of indices $\langle k, p\rangle$ is on $\mathfrak{b}$, add $[(c = c)]^k_p$ to the end of $\mathfrak{b}$ for all constants c on $\mathfrak{b}$.

IV If $[\psi]^j_m$ is $[(c = d)]^k_q$, for any constants c and d, and $[\varphi(c)]^k_p$ occurs on $\mathfrak{b}$, add $[\varphi(d)]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it).

V If $[\psi]^j_m$ is $[\neg\neg\chi]^k_p$, add $[\chi]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it).

VI If $[\psi]^j_m$ is $[(\zeta\wedge\chi)]^k_p$, add both $[\zeta]^k_p$ and $[\chi]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain them — analogously for the other conjunctive cases).

VII If $[\psi]^j_m$ is $[(\zeta\vee\chi)]^k_p$, split the end of $\mathfrak{b}$ into $\mathfrak{b}_1$ and $\mathfrak{b}_2$, adding $[\zeta]^k_p$ and $[\chi]^k_p$ to the end of $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively (if it does not already contain them — analogously for the other disjunctive cases).

VIII If $[\psi]^j_m$ is $[\Diamond\chi]^k_p$, take the first lower numeric index, q, which is new to $\mathfrak{b}$, and add $[\chi]^k_q$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\Box$).

IX If $[\psi]^j_m$ is $[\Box\chi]^k_p$, add every doubly-indexed formula $[\chi]^k_q$ to the end of $\mathfrak{b}$ for every index q occurring on the branch (if it does not already contain it — analogously for $\neg\Diamond$).

X If $[\psi]^j_m$ is $[\odot\chi]^k_p$, add $[\chi]^0_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\odot$).

XI If $[\psi]^j_m$ is $[\otimes\chi]^k_p$, add $[\chi]^p_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\otimes$).

XII If $[\psi]^j_m$ is $[\mathcal{A}\chi]^k_p$, add $[\chi]^k_k$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\mathcal{A}$).

XIII If $[\psi]^j_m$ is $[\mathcal{S}\chi]^k_p$, take the first upper numeral index, l, which is new to $\mathfrak{b}$, and add $[\chi]^l_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $\neg\mathcal{F}$).

XIV If $[\psi]^j_m$ is $[\mathcal{F}\chi]^k_p$, add every doubly-indexed formula $[\chi]^l_p$ to the end of $\mathfrak{b}$ for every index l occurring on the branch (if it does not already contain it — analogously for $\neg\mathcal{S}$).

XV If $[\psi]^j_m$ is $[(\exists x)\chi]^k_p$, take the first constant, c, not appearing on $\mathfrak{b}$, and add $[\chi[c/x]]^k_p$ to the end of $\mathfrak{b}$ (if it does not already contain it — analogously for $(\neg\forall x)$).

XVI If $[\psi]^j_m$ is $[(\forall x)\chi]^k_p$, add every doubly-indexed formula $[\chi[c/x]]^k_p$ to the end of $\mathfrak{b}$ such that c is among the first countably many individual constants available (if it does not already contain it — analogously for $(\neg\exists x)$).

Once this procedure is completed for the leftmost highest point on the 2D-tableau, repeat it for the next highest point closest from it until the rightmost highest point. This is where stage $n+1$ is concluded. Now there are three possible cases to consider. Either the systematic procedure resulted in a closed 2D-tableau, thereby producing a proof of $[\varphi]^i_n$; the procedure terminated, producing an open branch; the procedure does not terminate, producing a possibly infinite open branch.

Completeness for $\mathbf{S5}_{2D}$. We show how to construct a countermodel such that, if $\mathfrak{b}$ is any complete open branch of a 2D-tableau, finite or not, then $\mathfrak{b}$ is satisfiable, where $\mathfrak{b}$ is a *complete* open branch of a 2D-tableau $\mathcal{T}$ just in case every rule that can possibly be applied to it has been applied. In order to construct a countermodel we use the same procedure as in the counterexamples above.

Definition A.3 (Equivalence class) Let $\approx$ be an equivalence relation over the set of constants in $\mathcal{L}_{2D}$ such that $c_0 \approx c_1$ iff '$[c_0 = c_1]^i_n$' occurs on $\mathfrak{b}$ for any pair of indices $\langle i, n\rangle$. Moreover, for any c, let $[c]$ denote the equivalence class of c relative to $\approx$.

Definition A.4 (Model) For the countermodel, if k is any single index occurring on $\mathfrak{b}$, let $Z = \{k \mid k \in \mathfrak{b}\}$, and $W = Z \times Z$. Let $\langle 0, 0\rangle$ be a distinguished element of W. For every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_\Box \langle i, m\rangle$, and for every $\langle i, n\rangle, \langle j, n\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_\mathcal{F} \langle j, n\rangle$. Next we define C as the set of all constants occurring on $\mathfrak{b}$, and $\mathscr{D} = \{[c] \mid c \in C\}$. Furthermore, set $V(c, \langle i, n\rangle) = [c]$ to each $c \in C$, and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n\rangle) = \{\langle [c_0], ..., [c_n]\rangle : R(c_0, ..., c_n)^i_n \text{ occurs on } \mathfrak{b}\}$. Given both rigidity conditions, for every constant symbol c and $\langle i, n\rangle, \langle j, m\rangle \in W$ we have $V(c, \langle i, n\rangle) = V(c, \langle j, m\rangle)$. In case φ is atomic, if $[\varphi]^i_n$ occurs on $\mathfrak{b}$ then $\mathcal{M}^i_n \vDash \varphi$, otherwise set $\mathcal{M}^i_n \nvDash \varphi$.[58] This is enough to define a constant domain 2D-centered model $\mathcal{M} = \langle W, \langle 0, 0\rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$.

Lemma A.3 (Truth lemma) *Let* $\mathfrak{b}$ *be a complete open branch of a 2D-tableau, and* $\mathcal{M}$ *be a constant domain 2D-centered model,* $\mathcal{M} = \langle \mathrm{W}, \langle 0, 0\rangle, \mathcal{R}_\Box, \mathcal{R}_\mathcal{F}, \mathscr{D}, V\rangle$. *For every doubly-indexed formula,* $[\varphi]^i_n$,

$$[\varphi]^i_n \text{ occurs on } \mathfrak{b} \Leftrightarrow \mathcal{M}^i_n \vDash \varphi$$

Proof. By induction on φ. We only consider the relevant two-dimensional operators.

Let φ be $\mathcal{A}\psi$. If $[\mathcal{A}\psi]^i_n$ occurs on $\mathfrak{b}$, then $[\psi]^i_i$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^i_i \vDash \psi$, whence $\mathcal{M}^i_n \vDash \mathcal{A}\psi$, by the truth clause for $\mathcal{A}$ (the argument for $\neg\mathcal{A}\psi$ is analogous).

Let φ be $\odot\psi$. If $[\odot\psi]^i_n$ occurs on $\mathfrak{b}$, then $[\psi]^0_n$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^0_n \vDash \psi$, whence $\mathcal{M}^i_n \vDash \odot\psi$, by the truth clause for $\odot$ (analogously for $\neg \odot \psi$).

Let φ be $\otimes\psi$. If $[\otimes\psi]^i_n$ occurs on $\mathfrak{b}$, then $[\psi]^n_n$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^n_n \vDash \psi$, whence $\mathcal{M}^i_n \vDash \otimes\psi$, by the truth clause for $\otimes$ (analogously for $\neg \otimes \psi$).

Let φ be $\mathcal{F}\psi$. If $[\mathcal{F}\psi]^i_n$ occurs on $\mathfrak{b}$, then for every j on $\mathfrak{b}$, $[\psi]^j_n$ is also on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}^j_n \vDash \psi$ for every $\langle i, n\rangle \mathcal{R}_\mathcal{F} \langle j, n\rangle$, whence $\mathcal{M}^i_n \vDash \mathcal{F}\psi$, by the truth clause for $\mathcal{F}$. On the other hand, let φ be $\neg\mathcal{S}\psi$. If $[\neg\mathcal{S}\psi]^i_n$ occurs on $\mathfrak{b}$, then for every j on $\mathfrak{b}$, $[\neg\psi]^j_n$ also occurs on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}^j_n \nvDash \psi$ for every $\langle i, n\rangle \mathcal{R}_\mathcal{F} \langle j, n\rangle$, whence $\mathcal{M}^i_n \nvDash \mathcal{S}\psi$, by the truth clause for $\mathcal{S}$.

Let φ be $\mathcal{S}\psi$. If $[\mathcal{S}\psi]^i_n$ occurs on $\mathfrak{b}$, then for some j on $\mathfrak{b}$, $[\psi]^j_n$ is also on $\mathfrak{b}$. By I.H. and construction, $\mathcal{M}^j_n \vDash \psi$ for some $\langle i, n\rangle \mathcal{R}_\mathcal{F} \langle j, n\rangle$, whence $\mathcal{M}^i_n \vDash \mathcal{S}\psi$, by the truth clause for $\mathcal{S}$.

[58] Of course, we assume the models are constrained by (Neutrality).

On the other hand, let φ be $\neg\mathcal{F}\psi$. If $[\neg\mathcal{F}\psi]^i_n$ occurs on $\mathfrak{b}$, then for some j on $\mathfrak{b}$, $[\neg\psi]^j_n$ is also on $\mathfrak{b}$. By I.H., $\mathcal{M}^j_n \nvDash \psi$ for some $\langle i,n\rangle\mathcal{R}_{\mathcal{F}}\langle j,n\rangle$, whence $\mathcal{M}^i_n \nvDash \mathcal{F}\psi$, by the truth clause for $\mathcal{F}$. □

Theorem A.2 (Completeness) *If φ is valid, then φ has a 2D-tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a 2D-tableau proof, in which case any 2D-tableau $\mathcal{T}$ beginning with $[\neg\varphi]^0_0$ remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]^0_0$ occurs on $\mathfrak{b}$. By Lemma A.3, there is a constant domain 2D-centered model, $\mathcal{M} = \langle W, \langle 0,0\rangle, \mathcal{R}_\Box, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, such that $\mathcal{M}^0_0 \nvDash \varphi$. Consequently, φ is not valid. □

B Soundness and completeness theorems for general and diagonal tableaux

It might be useful to indicate how the soundness and completeness theorems for local 2D-tableaux can be modified for general and diagonal 2D-tableaux. The only differences concern the main theorems for soundness and completeness, namely, theorems A.1 and A.2. In what follows we show how they are adjusted for general 2D-tableaux.

Theorem B.1 (Soundness for general 2D-tableaux) *If φ has a general 2D-tableau proof, then φ is valid.*

Proof. Suppose φ has a general 2D-tableau proof, in which case there is a closed general 2D-tableau, $\mathcal{T}$, which begins with $[\neg\varphi]^n_m$, where $n \neq m$, and both n and m are different from '0'. For a contradiction, assume that φ is not generally valid. Thus, there is a 2D-centered model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_\Box, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, and a pair $\langle v, w\rangle \in W$, such that $\mathcal{M}^v_w \nvDash \varphi$. Let f be a function such that $f(n) = v$ and $f(m) = w$, in which case $\mathcal{M}^{f(n)}_{f(m)} \nvDash \varphi$. By Definition A.1, $\{[\neg\varphi]^n_m\}$ is satisfiable. Moreover, since its only branch is satisfiable, $\mathcal{T}$ is also satisfiable, and so is any general 2D-tableau extending it by way of the $\mathbf{S5}_{2D}$ rules, by Lemma A.2. Therefore, $\mathcal{T}$ is both closed and satisfiable, contradicting Lemma A.1, whence $\mathcal{M}^v_w \vDash \varphi$. □

Theorem B.2 (Completeness for general 2D-tableaux) *If φ is valid, then φ has a general 2D-tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a general 2D-tableau proof, in which case any general 2D-tableau $\mathcal{T}$ beginning with $[\neg\varphi]^n_m$, where $n \neq m$, both n and m are different from '0', and that remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]^n_m$ occurs on $\mathfrak{b}$. By Lemma A.3, there is a constant domain 2D-centered model, $\mathcal{M} = \langle W, \langle 0,0\rangle, \mathcal{R}_\Box, \mathcal{R}_{\mathcal{F}}, \mathscr{D}, V\rangle$, such that $\mathcal{M}^n_m \nvDash \varphi$. Consequently, φ is not generally valid. □

The arguments for diagonal 2D-tableaux are very similar, we just have to consider a sentence, φ, indexed by any pair, $\langle n, n\rangle$, where $n \neq 0$. Besides this, the arguments are essentially the same.

C Different systems

We show that sound and complete general 2D-tableau can be obtained for different systems extended by $\mathbf{S5}_{2D}$.

C.1 S5$\mathcal{A}$

Crossley and Humberstone offer the following axioms for **S5**$\mathcal{A}$ with general validity, besides the usual **S5** axioms:[59]

(A1) $\mathcal{A}(\varphi \supset \psi) \supset (\mathcal{A}\varphi \supset \mathcal{A}\psi)$

(A2) $\mathcal{A} \equiv \neg\mathcal{A}\neg\varphi$

(A3) $\Box\varphi \supset \mathcal{A}\varphi$

(A4) $\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi$.

Lemma C.1 ***S5***$_{2D}$ *with general validity extends* ***S5***$\mathcal{A}$ *with general validity.*

Proof. General 2D-tableau proof schemata can be given to prove axioms (A1)-(A4). We show (A3) and (A4), and leave (A1) and (A2) for the reader.

1. $[\neg(\Box\varphi \supset \mathcal{A}\varphi)]^0_1$
2. $[\Box\varphi]^0_1$
3. $[\neg\mathcal{A}\varphi]^0_1$
4. $[\neg\varphi]^0_0$
5. $[\varphi]^0_0$

$\times$

1. $[\neg(\mathcal{A}\varphi \supset \Box\mathcal{A}\varphi)]^0_1$
2. $[\mathcal{A}\varphi]^0_1$
3. $[\neg\Box\mathcal{A}\varphi]^0_1$
4. $[\neg\mathcal{A}\varphi]^0_2$
5. $[\neg\varphi]^0_0$
6. $[\varphi]^0_0$

$\times$

□

Corollary C.1 *Soundness and Completeness for* ***S5***$\mathcal{A}$ *with general validity.*

Proof. By Lemma C.1 and soundness and completeness for $\mathbf{S5}_{2D}$ general 2D-tableaux. □

Moreover, it can be shown that $\mathbf{S5}_{2D}$ extends **S5**$\mathcal{A}$ with local validity as well by proving the axiom schema $\mathcal{A}\varphi \supset \varphi$, which is straightforwardly done by local 2D-tableaux.

[59] In their original paper, Crossley and Humberstone use axioms with a rule of uniform substitution. In Davies and Humberstone (1980), axiom-schemata are offered, and a redundant axiom from Crossley and Humberstone (1977) is omitted. We follow Davies and Humberstone's presentation here.

C.2 S5$\mathcal{AF}$

In what follows we exhibit Davies and Humberstone's axiomatization of **S5**$\mathcal{AF}$. Besides the basis provided by **S5**$\mathcal{A}$, we have:

($\mathcal{F}$1) $\mathcal{F}(\varphi \supset \psi) \supset (\mathcal{F}\varphi \supset \mathcal{F}\psi)$

($\mathcal{F}$2) $\mathcal{F}\varphi \supset \varphi$

($\mathcal{F}$3) $\mathcal{F}\varphi \supset \mathcal{F}\mathcal{F}\varphi$

($\mathcal{F}$4) $\varphi \supset \mathcal{F}\neg\mathcal{F}\neg\varphi$

($\mathcal{F}$5) $\varphi \supset \mathcal{F}\varphi$, for any $\mathcal{A}$-free formula φ.

($\mathcal{F}$6) $\mathcal{F}\mathcal{A}\varphi \equiv \Box\varphi$, for any $\mathcal{A}$-free formula φ.

Lemma C.2 ***S5***$_{2D}$ *with general validity extends* ***S5***$\mathcal{AF}$ *with general validity.*

Proof. Again, general 2D-tableau proof schemata can be given for axioms ($\mathcal{F}$1)-($\mathcal{F}$6). □

Corollary C.2 *Soundness and Completeness for* ***S5****AF with general validity.*

Proof. By Lemma C.2 and soundness and completeness for **S5**$_{2D}$ general 2D-tableaux. □

Theorem C.1 *Soundness for* ***S5E***$_{2D}$*.*

Proof. The arguments are very similar to the corresponding proofs for **S5**$_{2D}$. We only need the following adjustments. The definition of satisfiability needs to be modified by taking into account $\mathcal{R}_{\mathcal{D}}$ rather than $\mathcal{R}_{\mathcal{F}}$, in which case for a constant domain 2D-centered model, $\mathcal{M} = \langle W, \langle w*, w*\rangle, \mathcal{R}_{\Box}, \mathcal{R}_{\mathcal{D}}, \mathscr{D}, V\rangle$, a set S of doubly-indexed formulas, and a function f from each individual index n in S to a possible world in Z, if the pairs of indices $\langle i, n\rangle$ and $\langle z, z\rangle$ are in S, then $\langle f(i), f(n)\rangle \mathcal{R}_{\mathcal{D}} \langle f(z), f(z)\rangle$. The rest of the definition remains the same. The soundness lemma corresponding to Lemma A.2 varies with respect to the latter in minor details: it does not need arguments for $\otimes$ and $\odot$, it contains clauses for $\mathcal{D}$ and $\mathcal{C}$ rather than $\mathcal{F}$ and $\mathcal{S}$, which will be very similar, besides not containing arguments for upper exchange. We leave the details for the reader. □

Theorem C.2 *Completeness for* ***S5E***$_{2D}$*.*

Proof. The arguments are, again, very similar to the ones for **S5**$_{2D}$. However, for the countermodel corresponding to Definition A.4, since we are not assuming rigidity with respect to the upper index, there needs to be some adjustments in our definitions. First, we need to modify Definition A.3 of equivalent classes. Let $\approx_{i,n}$ be an equivalence relation over the set of constants in $\mathcal{L}_{E2D}$ such that $c_0 \approx_{i,n} c_1$ iff '$[c_0 = c_1]^i_n$' occurs on $\mathfrak{b}$ for any constants c_0, c_1, and pair of indices $\langle i, n\rangle$. Moreover, for any c, let $[c]_{i,n}$ denote the equivalence class of c relative to $\approx_{i,n}$. Now, if k is any index occurring on $\mathfrak{b}$, let $Z = \{k \mid k \in \mathfrak{b}\}$, and $W = Z \times Z$. Let $\langle 0, 0\rangle$ be a distinguished element of W. For every $\langle i, n\rangle, \langle i, m\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\Box} \langle i, m\rangle$, and for every $\langle i, n\rangle, \langle z, z\rangle \in W$, set $\langle i, n\rangle \mathcal{R}_{\mathcal{D}} \langle z, z\rangle$. Let $\mathscr{D} = \{[c]_{i,n} \mid c, \langle i, n\rangle \text{ on } \mathfrak{b}\}$. Concerning rigidity, this time we only assume that for any constant symbol c and $\langle i, n\rangle, \langle i, m\rangle \in W$,

$V(c, \langle i, n\rangle) = V(c, \langle i, m\rangle)$, where $V(c, \langle i, n\rangle) = [c]_{i,n}$ and to each n-place predicate symbol, R, on $\mathfrak{b}$, let $V(R, \langle i, n\rangle) = \{\langle [c_0]_{i,n}, ..., [c_n]_{i,n}\rangle : R(c_0, ..., c_n)^i_n$ occurs on $\mathfrak{b}\}$. We do not assume (Neutrality) anymore. The rest of the definition remains the same as in Definition A.4. Furthermore, the truth lemma is basically the same except for, again, having clauses for $\mathcal{D}$ and $\mathcal{C}$ rather than $\mathcal{F}$ and $\mathcal{S}$, which are straightforward. □

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